DISPLACEMENT INTERFEROMETRY BY THE AID OF THE ACHROMATIC FRINGES

PART III

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PREFACE.

The present report is chiefly devoted to the investigation of methods of research in which displacement interferometry, conducted by the aid of the achromatics discussed in the preceding report, gives promise of fruitful applications. Thus, in Chapter I the method of measuring small angles hitherto suggested is given a practical test. The general theory of the subject in its bearing on the two possible methods is developed at some length and a variety of interferometer devices, with mirror, ocular, and collimator micrometers, are instanced. Unfortunately, it was not till after the end of these experiments that I detected the method of reducing the fringes to the smallest number possible, practically to a single fringe; otherwise the work would have been more satisfactory throughout.

As the achromatic fringes can not (in general) be found without first finding the corresponding spectrum fringes and, conversely, since for each type of spectrum fringes (direct or reversed) a corresponding group of achromatic fringes may be associated, I have devoted Chapter II to spectrum fringes differing in their manner of production. The endeavor here has been to obtain interferences from distant slender luminous objects, without the aid of a slit. Partially at least the work has succeeded, but not as far as I hoped. The experiments are very difficult.

The work in the third chapter was undertaken at the request of Prof. W. G. Cady, of Wesleyan University, in the endeavor to obtain the elastic constants of small bodies. The application of the displacement method proved at once to be astonishingly easy in a case where a degree of rough handling is inevitable; but there lurked in the elastic apparatus some discrepancy, both of viscosity and hysteresis, the nature of which escaped detection even after many attempts to locate its origin.

Chapter IV contains applications of the rectangular interferometer using achromatic fringes to geophysical problems. A method for the determination of the Newtonian constant is worked out. Again, the same interferometer is associated with the horizontal pendulum for the detection of small changes in the inclination of the earth's surface. Series of observations extending between January and August are recorded.

Finally, in the last chapter, I have investigated corresponding methods for the interferometry of vibrating systems. The luminosity of the achromatic fringes lends itself easily to this purpose and it was merely necessary to

design an appropriate vibrating telescope. To test the method, a study is made of the vibration of telephonic apparatus. For the first time I have obtained clear-cut interference vibration curves for two identical telephonic systems joined directly in series, while these forms subsided completely when the telephones were joined differentially. Such a system, from another point of view, is an electric dynamometer capable of appreciating an average alternating current well within a microampere. It could, moreover, be synchronized with an external impulse by aid of the Lissajous curves with the same accuracy as two tuning-forks.

CARL BARUS.

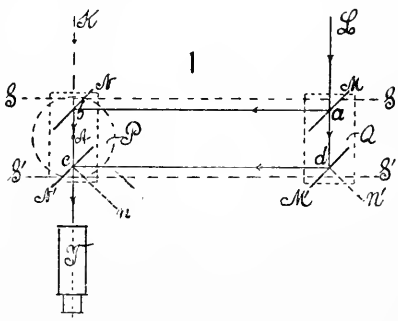
Providence, August 1918.

CHAPTER I.

THE DISPLACEMENT INTERFEROMETRY OF LONG DISTANCES.

1. Introduction.—Methods for the measurement of small angles and of long distances were broached at the end of the last report. It is the purpose of the present chapter to continue the work experimentally, with a view to further development. It will therefore be desirable to collect the useful equations in this place in relation to the form of apparatus to be adopted, as well as to deduce the consequences of these equations in relation to their bearing on displacement interferometry in general. Throughout the chapter the work is done chiefly with the aid of the achromatic fringe groups, as I have called them; but these, as a rule, can be found only by means of the spectrum fringes, wherefore the latter become of coördinate importance.

2. Apparatus.—This is an interferometer of the Jamin-Mach type (fig. 1) with four vertical plate mirrors, M,M', N,N', in parallel and at 45° to the horizontal beam of impinging white daylight L, from the country beyond. Three of these mirrors are half-silvered, viz, M,N,N', while M' is or may be opaque. The equal distances $ab=cd=b$ constitute the base-line (b) of the apparatus and the rectangle $abcd$ will be called the ray parallelogram. Its area is $2Rb$, where $ad=bc=2R$. Each of the mirrors must be provided with three adjustment screws for fine motion, so that the mirrors are each slightly revolvable about a vertical and a horizontal axis.



In case of the mirrors N, M', these screws must be convenient to manipulate (thumb-screws), for adjustment here will frequently be necessary. The opaque mirror M' is on a Fraunhofer micrometer slide with its screw in the direction of the normal n' to M' and adapted to readings of at least 10^{-4} cm.

In the figure S, S' may be regarded as slides of a lathe-bed on which the carriages P and Q carrying the mirrors may move longitudinally. Any distance $ab=cd$ is thus available for a base-line.

The telescope is at T and receives both the light from L after two reflections from the paired mirrors and the direct light from K through the half-silvers. It is desirable that the two beams from K and L be of about equal intensity.

The pair of mirrors N and N' is on a vertical axis A, so that it may be rotated as a whole. The amount of rotation may be read off either directly on a divided circle corresponding to the axis A or indirectly by the displacement of M' on the screw at n' in relation to the observed sweep of interference fringes.

Thus there are three objects seen in the telescope: the direct landscape from K, the reflected landscape from L, and the achromatic interference fringes due to the partial beams *abc* and *adc* when the apparatus is in adjustment. To find the latter the method pursued in the preceding report suffices. It is first necessary to find the spectrum interferences when L is a beam of intense white light from a collimator and fine slit and the telescope is provided with a direct-vision prism. These are to be centered by moving the micrometer at M' and adjusting the pair of mirrors $M'N$. The spectroscope is now removed (swiveled out) and the slit broadened, whereupon the intense achromatic fringes will appear covering the position of the image of the originally fine slit. If not vertical, the fringes may be made so by further slightly rotating M' and N on a horizontal axis, in the absence of compensators.

The collimator at L is then removed and the fringes will be found superposed on the foreground. If they are not bright enough from the light of the landscape, they may be given any intensity by reflecting (by way of *adc* and *abc*) a narrow horizontal strip of skylight from white paper into the telescope from L. Intense fringes will be seen transverse to the strip.

3. Rigorous equations.—In addition to the sides of the ray parallelogram b (base) and $2R$ (R radius of rotation) we shall have to consider the following angles or angular increments: $\Delta\alpha$ the angular rotation of the paired mirrors, $\Delta\theta$ the corresponding angular displacement of the fringes, ΔN the linear displacement of the micrometer mirror in a direction normal to its face, and $\Delta\varphi$ the angle subtended by two consecutive fringes. If n is the order of the fringe, we may write

(1) $$\Delta\varphi = \frac{\Delta\theta}{\Delta n}$$

Moreover, if i is the angle of incidence (45°) of L at the mirrors and λ the wave-length in question,

(2) $$\frac{2 \cos i \, \Delta N}{\Delta n} = \lambda$$

or on substitution,

(3) $$\frac{\Delta\theta}{\Delta N} = \frac{\Delta\varphi}{\Delta N/\Delta n} = \frac{2\Delta\varphi \cos i}{\lambda}$$

Moreover, if s is the angle at the apex of the distance triangle on the base b,

(4) $$\Delta s = 2\Delta\alpha$$

(5) $$\Delta\alpha = \frac{\Delta N \cos i}{R}$$

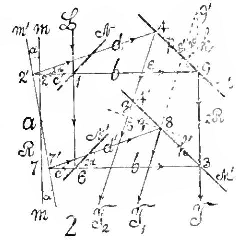
And since the distance $d = b/2s = b/2\,\Delta\alpha$, from (5),

(6) $$d = \frac{bR}{2\Delta N \cos i} = \frac{F}{\Delta N}$$

so that the sensitiveness is from (6),

$$\delta d = \frac{2d^2 \cos i}{bR}\, \delta(\Delta N) = \frac{\delta \Delta N}{Fd^2} \tag{7}$$

(a) It will next be desirable to deduce the above fundamental equations more rigorously than has thus far been done. Figure 2 is supplied for this purpose, and represents the more sensitive case where, in addition to the mirrors $MM,'$ NN' (all but M being necessarily half-silvers), there is an auxiliary mirror, mm, capable of rotation (angle α) about a vertical axis a. The mirrors, M - - - - N', in their original position, are conveniently at 45° to the rays of light, while mm is normal to them. Light arriving at L is thus separated by the half-silver N at 1, into the two components 1, 2, 1, 9, 3, T and 1, 6, 7, 6, 3, T, interfering in the telescope at T.



When mm is rotated over a small angle α, these paths are modified to 1, 2, 2′, 4, 4′, 5, T_2 and 1, 6, 7, 8′, T_1 and T_2 enter the telescope in parallel and produce interferences visible in the principal focal plane, provided the rays T_1 and T_2 are not too far apart, in practice not more than 1 or 2 mm. Interference fringes therefore will always disappear if the angle α is excessive, but the limits are adequately wide for all purposes. The essential constants of the apparatus are to be:

$(9, 1) = (6, 3) = b \qquad (1, 2) = (6, 7) = C \qquad (9, 3) = (1, 6) = (2, 7) = 2\,R$

R being the radius of rotation.

Where the mirror mm is rotated to $m'm'$ over the angle α the new upper path will be:

$$C + R \tan \alpha + d + e + g$$

where $(2'\ 4) = d$, $(4, 4') = e$, $(4,'5) = g$, the plane $(8, 5) = q$ normal to T_1, and T_2 being the final wave-front. The lower path is similarly

$$2\,R + (C\text{-}R \tan \alpha) + d'$$

to the same wave-front (8, 5), where $(7', 8) = d'$. Hence (apart from glass paths, which are preferably treated separately) the path-difference $n\lambda$ (n being the order of interference) should be

$$n\lambda = 2\,R\,(\tan \alpha - 1) + d - d' + e + g$$

The figure, in view of the laws of reflection, then gives us in succession

$$\begin{aligned}
d &= (b + c + R \tan \alpha)\ /(\cos 2\,\alpha + \sin 2\,\alpha) \\
d' &= (b + c - R \tan \alpha)\ /(\cos 2\,\alpha + \sin 2\,\alpha) \\
e &= 2\,R\ /(\cos 2\,\alpha + \sin 2\,\alpha) \\
g &= 2\,R \sin 2\alpha\,(1 + \tan \alpha)\,(\cos 2\,\alpha - \sin 2\,\alpha)/(\cos 2\,\alpha + \sin 2\,\alpha) \\
q &= 2\,R \sin 2\,\alpha\,(1 + \tan \alpha)
\end{aligned}$$

To obtain g it is sufficient to treat the similar triangles (3, 8, 9′) and (9, 8′, 9′), where $h=(9,4)$, $h'=(3,8)$, $k=(9,9')$, $l=(9,8')$ may be found in succession, as the normal distance between the mirrors M and M' is $R\sqrt{2}$, so that finally

$$g=(h\text{-}l)\sin(45^\circ-2\alpha) \qquad q=(h\text{-}l)\cos(45^\circ-2\alpha)$$

If these quantities are introduced into the above equation for $n\lambda$, we may obtain, after some reduction,

$$n\lambda=4R\sin\alpha\,(\cos\alpha-\sin\alpha)$$

Since $n\lambda=2\,\Delta N\cos i$, ΔN being the normal displacement of the mirror M' and $i=45^\circ$, the corresponding equation to the second order of small quantities, α, is

$$\frac{\Delta N}{\Delta\alpha}=\frac{2R}{\cos i}(\cos\alpha-\sin\alpha)=2\sqrt{2}R(1-\alpha-\alpha^2/2)$$

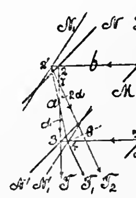
If α is sufficiently small, the coefficient is simply $2R/\cos i$ as used above.

There remain the glass paths which for the rays d and d' are compensated. Additionally the upper ray has a glass path (3) displaced to (4′), The lower ray has the fixed path at (1), and this is equal to the other at (1), since the angles are 45°. Thus the variable part of the glass paths at (3) to (4′) is uncompensated and the angle of incidence changes from 45° to $45^\circ-2\alpha$. The reflecting sides of the plates are silvered. Hence

$$e\,(\sin i-\cos i\tan r)2\Delta\alpha=\sqrt{2}\,(1-\tan r)e$$

must be added to the equation.

(*b*) The second case, figure 3, in which the auxiliary mirror of the preceding apparatus is omitted, is curiously enough inherently simpler. MM', NN' are mirrors (half-silvered at (1) and (3), and the two latter on a vertical axis a, and rigidly joined by the rail (2, 3). The mirrors being preferably at 45°, the component rays are 1, 2, 3, T and 1, 5, 3, T, the mirror M' being on a micrometer with the screw normal to the face. The ray parallelogram is made up as before of $(1,2)=b=(3,5)$ and $(1,5)=2R=(2,3)$. When the rail (2, 3) is rotated over an angle α, the mirrors take the position N_1 and N'_1, at an angle α to their prior position, and the angle of incidence is now $45^\circ-\alpha$. The new paths, if (4, 6) is the final wave-front, are thus (1, 2, 2′, 6, T_2) and (1, 5, 4, T_1). The rays T_1 and T_2 are parallel and interfere in the telescope. Hence the path-difference introduced by rotation is (n being the order of interference)



3

$$n\lambda=b+R\tan\alpha+\frac{2R}{\cos\alpha}\cos\alpha-(2R+b-R\tan\alpha)$$

or

$$n\lambda=2R\tan\alpha$$

for the triangle (a, 7, 2′) is isosceles and its acute angles each α.

The rays T_1 and T_2 have now separated and the amount (4, 6) is also $2R\tan\alpha$. When this exceeds a few millimeters the interferences vanish.

A correction must, however, be applied, since in the practical apparatus the mirrors rotate at a fixed distance apart, $2R$. Hence the mirror N_1 must be displaced toward the right (shortening the path) by the normal distance.

$$e=(R/\cos\alpha-R)\cos 45°$$

and the mirror N_1' toward the left by the same amount. The path-difference introduced is thus a decrement and is twice the $2\,e\cos(45°-\alpha)$ of each mirror. Thus the total correction to be subtracted from the equation is

$$\frac{4R}{\cos\alpha}(1-\cos\alpha)\frac{\sqrt{2}}{2}(\sin\alpha+\cos\alpha)\frac{\sqrt{2}}{2}=2R(1-\cos\alpha)(1+\tan\alpha)$$

Hence the equation becomes after reduction

$$n\lambda=2R\,(\tan\alpha-1-\tan\alpha+\sin\alpha+\cos\alpha)$$

or

$$n\lambda=2R\,(\sin\alpha+\cos\alpha-1)$$

To the second order of small quantities, if $i=45°$ is the angle of incidence and ΔN the normal displacement of M',

$$\frac{\Delta N}{\Delta\alpha}=R\frac{1+\Delta\alpha/2}{\cos i}$$

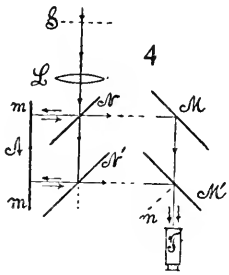
As all the mirrors receive the light on their silvered sides, M originally compensates N if the mirrors are identical in thickness and glass. But the transmission at (3) varies as the angle of incidence changes from $i=45°$ to $45°-\alpha$. The glass path here decreases by

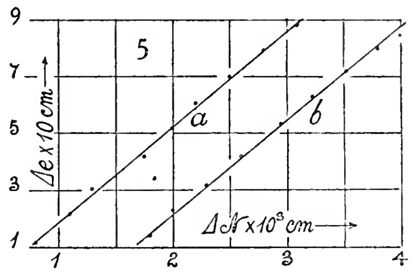
$$e\,(\sin i-\cos i\tan r)\,\Delta\alpha$$

where e is the plate thickness, r the angle of refraction. The path-difference as above reckoned has thus been increased by this amount, and this quantity is to be added to the right-hand member. The effect will not usually exceed a small percentage of the air-path-difference and the ratio is the same as above.

4. Ocular micrometer.—It has been stated that the motion of the fringes across the field of the telescope (T, fig. 4) is astonishingly swift; hence it is often desirable to insert a micrometer here, as the displacement of fringes can thus be much more accurately and easily measured than at the micrometer along the normal n of the opaque mirror M of the interferometer. If the latter is of the type using an auxiliary mirror (mm, fig. 4) the fringes may even be established of a size to correspond with the ocular micrometer by rotating the auxiliary mirror; but this is not usually necessary. A good ocular plate micrometer was at hand, dividing the width of field (about 1 cm.) into 100 parts, the divisions being 0.1 mm. One-tenth of this is easily estimated by the eye in view of the eye lens. The light from the collimator SL should completely fill the field, a condition which may be fulfilled by

suitably placing the former, or modifying its objective L. After completing such preliminary adjustments with the fringes, made very sharp, and the ocular scale equally so, this is to be placed at right angles to the fringes. Let Δe denote their displacement measured in centimeters on the ocular scale and ΔN (cm.) the displacement of the opaque mirror M of the interferometer. The question is whether Δe and ΔN are nearly enough proportional quantities for practical purposes. A number of such standardizations





were carried out throughout 1 cm. of Δe, two of which are shown in detail in figure 5. The fluctuation of data is due to air-currents across the interferometer. It was not easy to obviate these, and it was not thought necessary for the present purposes. Otherwise the data would have been smooth. There is no doubt that a linear relation may be assumed. In curve *a* the readings of the interferometer micrometer increase, in curve *b* they decrease. If the means be taken from doublets far apart the ratios are

$$(a)\quad \Delta N/\Delta e = 0.00310 \qquad\qquad (b)\quad \Delta N/\Delta e = 0.00310$$

and they happen to coincide. Thus Δe is 323 times as large as ΔN and correspondingly easy to measure. The impossibility of setting the micrometer for ΔN accurately enough, since it is graduated to only 5×10^{-5} cm., is completely obviated in Δe. Moreover, as $2\Delta N \cos i = \lambda$ (i being the angle of incidence, 45°, and λ the mean wave-length), we now have

$$0.0061\, \Delta e \cos i = \lambda$$

so that the fringe displacement

$$\Delta e = \frac{6\times10^{-5}}{6.1\times10^{-3}\times0.71} = 0.014 \text{ cm.}$$

measured on the ocular micrometer, corresponds to the wave-length of light in the interferometer measurements. This is more than one scale part. There is, however, no difficulty in making the fringes larger and obtaining a much more sensitive apparatus in proportion. The achromatic fringes, moreover, when properly produced, contain a distinctive central black line compatible with the measurement of 0.1 scale part, as here given; i. e., measurement to a few millionths of a centimeter are thus easily feasible under proper surroundings. The apparatus will be used below.

If $\Delta\varphi$, the angular fringe breadth, is given, $\Delta\theta/\Delta N$ may be computed from the above equation, Chapter I, No. 1, since

$$\frac{\Delta\theta}{\Delta N}=\frac{2\Delta\varphi\cos i}{\lambda}$$

or

$$\frac{\Delta e}{\Delta N}=\frac{2L\Delta\varphi\cos i}{\lambda}\text{ and }Ld\varphi=\frac{\lambda}{2\cos i\,\Delta N/Ne}$$

as both $\Delta\theta$ and Δe have the same radius; i. e., the length $L=19.5$ cm. of the telescope. Hence the fringe breadth in centimeters is, if $\lambda=6\text{x}10^{-5}$ cm., $i=45°$ and $10^3\Delta N/\Delta e=3.1$

$$L\Delta\varphi=6\times10^{-5}/(2\times0.071\times3.1\times10^{-3})=0.014\text{ cm.}$$

the value actually observed. Thus, if $\Delta\varphi$ is given or measured, $\Delta e/\Delta N$ may be deduced.

The question finally to be determined is thus the value and the meaning of the fringe breadth $\Delta\varphi$. Since $2\Delta N\cos i=n\lambda$, if $\Delta N=\Delta N_0$ is constant and also λ, we may then write

(8) $$\Delta\varphi=\frac{di}{dn}=-\frac{\lambda}{2\Delta N_0\sin i}$$

Furthermore,

(9) $$\Delta\theta=n\Delta\varphi=-\frac{\Delta N}{\Delta N_0}\cot i$$

if $n\lambda$ is replaced by its value and ΔN is small compared with ΔN_0. If it is not, since ΔN_0 involves ΔN, we must state the case thus:

$$-\Delta\theta=\cot i\int_{\Delta N_0}^{\Delta N_0+\Delta N}dN/\Delta N_0=\cot i\log\frac{\Delta N_0+\Delta N}{\Delta N_0}$$

or, on expanding the natural logarithm,

(10) $$-\Delta\theta=\cot i\left(\frac{\Delta N}{\Delta N_0}-\frac{1}{2}\left(\frac{\Delta N}{\Delta N_0}\right)^2+\ldots.\right)$$

and

$$-\Delta e=L\Delta\theta$$

In the above measurements

$$\Delta\varphi=\frac{L\Delta\varphi}{L}=\frac{0.014}{19.5}=7.2\times10^{-4}$$

whence (apart from signs)

$$\Delta N_0=\frac{\lambda}{2\Delta\varphi\sin i}=0.06\text{ cm., nearly}$$

whereas the maximun displacement ΔN throughout the whole series (or field width in fig. 5) does not exceed $\Delta N=5\times10^{-3}$ cm. Hence $(\Delta N/\Delta N_0)^2/2$ may here be neglected to about 1/300 and (again apart from signs) since $i=45°$,

$$\Delta e=L\Delta N/\Delta N_0=19.5\frac{\Delta N}{0.06}=325\Delta N$$

as it should be; i. e., the relation of Δe and ΔN is practically linear, if the displacement ΔN is not excessive or goes beyond the field width.

As the determination of ΔN_0 is inconvenient, we thus come back to the practical equation (3) already used, or

$$\Delta N/\Delta\theta = \lambda/2\Delta\varphi \cos i$$

or if $Ld\varphi = \delta e$ and Δe and δe are the displacement and the fringe breadth measured on the same ocular micrometer,

(11) $$\frac{\Delta N}{\Delta e} = \frac{\lambda}{2\delta e \cos i}$$

With this deduction, equations (5), (6), (7) now take on the new form in terms of the fringe breadth δe and the fringe displacement Δe, which it is well to record here;

(12) $$\Delta\alpha = \frac{\lambda}{2R\delta e}\Delta e$$

(13) $$d = \frac{bR\delta e}{\lambda}\frac{1}{\Delta e}$$

(14) $$\delta d = \frac{d^2\lambda}{bR\delta e}\delta(\Delta e)$$

If into the last equation we insert such values as are easily obtained and will be used in the sequel, viz, d = kilometer = 10^5 cm.; $b = 200$ cm.; $R = 10$ cm.; $\delta e = 0.015$ cm.; $\delta(\Delta e) = 0.001$ cm. (1/10 scale part of ocular micrometer); then

$$\delta d = \frac{10^{10}\times 6\times 10^{-5}\times 10^{-3}}{200\times 10\times 15\times 10^{-3}} = 20 \text{ cm.}$$

that is, an object should be located at a distance of a kilometer to 20 cm. If there is sufficient light, however, fringes may be easily made larger than 0.015 cm. in breadth, certainly 3 to 5 times, in which case δd would be correspondingly smaller. With ordinary daylight, however, $10^3\delta e = 10$ to 20 cm. should not be exceeded.

Results similar to the preceding may be obtained if the glass paths of the interferometer are alone considered.

If μ is the index of refraction and e is the effective thickness of the plates, i. e., the difference of effective thickness of M and N and of compensators (fig. 1) through which the beams pass, we may write as in the colors of thin plates (since the beam passes the plate but once)

(8) $$n\lambda = e\mu \cos r - 2N \cos i$$

if r is the angle of refraction corresponding to the angle of incidence i. If n, i, r alone vary, while e, λ, N are fixed (i. e., if the eye travels through the field of the telescope from left to right), $\Delta\varphi = di/dn$, and since $\sin i = \mu \sin r$,

(9) $$\Delta\varphi = \frac{di}{dn} = \frac{\lambda}{2N \sin i - e \tan r \cos i}$$

so that $\Delta\varphi$ depends inversely on e and N. When the spectrum ellipses are centered, $N=N_0$, a condition necessary for the occurrence of achromatic fringes, where

$$(10') \qquad 2\,N=2\,N_0=e\,(\mu\cos r+2\,B/\lambda^2\cos r)/\cos i$$

if $\lambda d\mu/d\lambda=2\,B/\lambda^2$ is adequate. Equation (10) may now be inserted in equation (9) and the coefficient of e, viz, the long parenthesis containing circular functions evaluated for $i=45°$, $\lambda=60\times10^{-6}$, $\mu=1.55$, $2\,B/\lambda^2=0.026$. Its value is slightly greater than 1. Hence we may write approximately but with much greater convenience,

$$(11') \qquad \Delta\varphi=\lambda/e$$

We thus obtain the breadth of fringes for different values of the micrometer e, roughly. However, it would be possible to compute the accurate value of i, fulfilling the condition (10′).

These equations placed in the above (3), (5), (6) give in succession

$$(12') \qquad \Delta\theta=\frac{2\Delta\varphi\,\Delta N\cos i}{\lambda}=\frac{2\cos i}{e}\Delta N$$

$$(13') \qquad \Delta\theta=\frac{2R}{\lambda}\Delta\varphi\Delta\alpha\qquad=\frac{2R}{e}\Delta\alpha$$

$$(14') \qquad d=\frac{bR}{2\,\Delta N\cos i}\qquad=\frac{bR}{e\Delta\theta}$$

so that the measurement of the long distance d depends ultimately on the area $2bR$ of the ray parallelogram, the differential thickness of paired glass plates e, and the displacement $\Delta\theta$ of achromatic fringes.

From equation (14) we obtain the sensitiveness by differentiation, or

$$(15') \qquad -\delta d=\frac{d^2e}{bR}\,\delta(\Delta\theta)$$

Let the angle $\Delta\theta$ or its variation be measured in a telescope of length L and provided with an ocular micrometer, so that the angle $\Delta\theta=x/L$, x being the linear magnitude measured on this micrometer. Hence

$$(16') \qquad \delta d=\frac{d^2e}{bRL}\,\delta x$$

It will be noticed that equations (12′) to (14′) are the same as equations (12) to (14), if e in the latter is replaced by $\lambda/\Delta\varphi$ (equation 11′), since $\Delta e=L\Delta\theta$ and $\delta e=L\Delta\varphi$.

5. Collimator micrometer.—It is obvious that the ocular micrometer may be an image of a scale, S (fig 4), placed in the wide slit of the collimator SL. The method is even more sensitive than the last and has the additional advantage that the telescope, containing no fiducial lines, may be shifted at pleasure. This is a great convenience when promiscuous experiments are contemplated. It was found best, in case of the preceding apparatus

without ocular micrometer, to remove the slit altogether (fig. 4) and install a glass scale, S, about 2.5 cm. long and divided into 100 parts, in its place. A millimeter scale is usually too coarse. The scale S must be so inclined to the horizontal that its image is normal to the fringes in the telescope. No difficulty was found either as to clearness or precision with this adjustment. Measurements were made, from which (ΔN is the normal displacement at the mirror M, fig. 4) and $\Delta\epsilon$ the displacement of fringes along the imaged micrometer in the ocular)

$$\frac{\Delta N}{\Delta\epsilon}=0.00101$$

was found for the tenth-millimeter fringes considered in the preceding paragraph. Here $\Delta\epsilon$ is therefore nearly 1,000 times larger than ΔN. We may thus write $0.00202\ \Delta\epsilon \cos i=\lambda$, so that

$$\Delta\epsilon=\frac{6\times 10^{-5}}{2\times 10^{-3}\times 0.71}=0.042 \text{ cm.}$$

or nearly half a millimeter, corresponds to the wave-length of light in the interferometer measurements. With larger fringes the precision may be enhanced and sharp achromatic lines are available as before.

If we denote by $\delta\epsilon$ and $\Delta\epsilon$ the fringe breadth and the displacement of fringes in case of the collimator micrometer (imaged in the ocular) and by $\delta\epsilon$ and Δe the corresponding quantities in case of the plate ocular micrometer, and if F and f be the principal focal distances of the objectives of the collimator and of the telescope (fig. 4), the relations are obviously

$$\frac{\delta e}{\delta\epsilon}=\frac{\Delta e}{\Delta\epsilon}=\frac{f}{F}$$

so that equations (10) to (14) are equally true when e is replaced by ϵ; but

$$\Delta N/\Delta e=\frac{F}{f}\frac{\Delta N}{\Delta\epsilon}$$

Hence $\Delta N/\Delta e$ is larger than $\Delta N/\Delta\epsilon$ in the ratio of F/f as actually found; but the data of the collimator micrometer at the slit are at once admissible in computing.

6. Half-silvered films.—The case of regular repetition of the achromatic phenomenon in case of white light, with the ghosts successively fainter, have frequently been mentioned. To possibly interpret this phenomenon a pair of half-silver plates were pressed together on the half-silver sides, so as to inclose a thin film of air between them. This double plate was then put into the beam from the collimater normally, whereupon a succession of ghosts was seen in the telescope, whereas none appeared without the air film. Hence they must be produced by reflection. In one case three repetitions were seen on the left and two (much enlarged) on the right of the main interference fringes, usually at an angle to them and more or less curved. The

inclination of fringes was usually opposite on the two sides. In spite of differences in size the successive repetitions of fringes were nearly equidistant. Thus four at a distance of $\Delta e=0.075$ cm. were seen. At another part of the double half-silver, $\Delta e=0.085$ cm. was measured. Putting a steel clip on the plates to force them more closely together, Δe was reduced to 0.040 cm. Taking the clip off increased Δc. No ghosts occurred with a single half-silver plate (no air film). Again, the distance Δe varied with the angle of incidence of the plate pair with the collimated beam of light. Thus, in case of three vivid interference grids (two repetitions), if i is the angle of incidence, the measurements gave, when the plate was placed at different angles i (estimated)

$i=0°$	$i=30°–40°$	$i=50°–60°$	$i=70°–80°$
$e=0.15,0.48,0.87$	$0.22,0.48,0.79$	$0.30,0.49,0.70$	$0.29,0.39,0.49,0.65,0.80$

In the last case the number of ghosts increased, but they also grew more irregular and confused.

Hence the cause of this originally puzzling phenomenon must be some reflection on the two sides of an air film, or a glass film. If x is the normal thickness of the film, and i the angle of incidence, the direct rays and those twice reflected interfere with a path-difference of $2x \cos i$. To restore the fringes to the center the main micrometer would have to move over ΔN or annul a path-difference of

$$2\frac{\Delta N}{\Delta e}\Delta e \cos i$$

so that

$$x \cos i = (\Delta N/\Delta e)\Delta e \cos i = 0.0033\times 0.7\Delta e = 0.0023\Delta e$$

Thus in the above data the reproductions would be at

$\delta e=0.36$	0.28	0.20	0.80
$10^5 x=$ 83	83	82	82 cm.
$i=$ 0°	36°	54°	77°

provided the angles were such as here given, which was sufficiently near the case.

It is nevertheless difficult to surmise what reflections can occur in the earlier work and in the absence of a specially half-silvered biplate; for the effect of an air film between clear glass plates is only visible with difficulty.

A similar problem is the measurement of the index of refraction, μ, of a normal film of glass in terms of δe. The relation is here

$$((\mu-1)+2\ B/\lambda^2)\ d=2\ (\Delta N/\Delta e)\ \Delta e \cos i = 0.0047\ \Delta e$$

A film of mica $d=0.0050$ cm. thick was inserted between the mirrors of the interferometer and the displacement $\Delta e=$ 0.65 cm. read off. Hence, if $2B/\lambda^2=0.026$ roughly,

$$\mu = 1+0.0047\times 0.65/0.0050-0.026=1.58$$

This is slightly low, for the thickness d and the dispersion constant B are not adequately guaranteed; but the result is interesting, as it may be obtained instantly, either in terms of Δe or ΔN.

7. Direct observations.—The first experiments were made upon objects lying across the campus of Brown University, since these distances, though relatively small, were measurable. Unfortunately there were few available long clear stretches, and distances of two objects at $d = 9{,}990$ cm. and 3,060 cm. were used. They were not, of course, in the same straight line, so that the constant small angle between them must be borne in mind. The constants of the apparatus were also correspondingly small (as it was necessary to look out of a window), being $b = 51.8$ cm., $2R = 9.4$ cm., obtained by passing a beam of collimated sunlight through the system of mirrors and measuring the length and breadth of the ray parallelogram. The angle of incidence was $i = 45°$. Hence, since $d = bR/2 \cos i \; \Delta N = F/\Delta N$, the factor is

$$F = bR/2 \cos i = 172.2.$$

It is first necessary to get ΔN_0, the micrometer position for parallel rays, as the plate mirrors were common plate glass. This is done by inverting the equation, since $\Delta N = F/d$ and $\Delta N' = F/d'$ for the two objects. The results were $\Delta N = 0.01723$ cm., $\Delta N' = 0.05626$ cm. Hence the computed equivalent of the angle between the two objects is $\Delta N' - \Delta N = 0.03903$ cm. (computed). The value directly observed in the interferometer was $\Delta N' - \Delta N = 0.0389$ cm. (observed), a very satisfactory agreement. Here it is to be carefully noted that the fringes in the second measurement $\Delta N'$, must be placed in coincidence with the displaced image of the first object, not with the coincident images of the second object, in which case the micrometer reading ($\Delta N' = 0.0411$ cm.) would be too large, or with the actual direction of the first object (non-reflected beam K), in which case the reading ($\Delta N' = 0.0368$ cm.) would be too small; for the two objects are not in the same straight line. This is of course very important, even if the angles are small.

We now have

$$\Delta N - \Delta N_0 = 0.01723 \text{ cm.} \qquad \Delta N' - \Delta N_0 = 0.05626 \text{ cm.}$$

where $\Delta N = 0.0459$ and $\Delta N = 0.0839$ are the observed values. Thus $\Delta N_0 = 0.0277$ cm. and the practical equation now reads

$$d = bR/2 \cos i \; (\Delta N - \Delta N_0).$$

Inverting the operation, we thus find

First object, observed: $\Delta N = 0.0450$ cm., $d = 9{,}992$ cm. (computed), $d = 9{,}990$ cm. (observed).

Second object, observed 0.0339 cm., 3,063 cm. (computed), 3,060 cm. (observed).

The constant ΔN_0 is independent of the fatcor F. Hence the base b and R may be changed at pleasure, from the exceptionally small value 51.8 cm. here used. Finally, for this small base b of but about half a meter the micrometer play between $d = 10$ meters and infinity would be ($\Delta N - \Delta N_0 = F/d$)

$d = 10$ meters	$\Delta N = 0.172$ cm.
$d =$ kilometer,	$\Delta N = 0.0017$ cm.

and the sensitiveness δd if $\delta\ (\Delta N) = 10^{-4}$ cm. is the least micrometer reading at

$d = 10$ meters $\delta d = 0.58$ cm.
$d = 100$ meters 58 cm.
$d =$ kilometer 5,814 cm.

in view of the small base 52 cm. and small $R = 4.7$ cm. The observed data were well within this. Of course on using the fringes individually these results could be immensely improved.

The endeavor was now made to work at larger distances d and a larger base b. But the laboratory being surrounded almost on all sides by trees, it was found that only at one upper window was a distance prospect visible, and hence, since it was again necessary to look through a window obliquely, the base was restricted to but $b = 36.6$ cm., not much above a foot. Still, as the values of ΔN are proportional to b, the purpose of the experiments could be adequately carried out within a range of about a mile, using the distant hill or horizon for ΔN_0 corresponding to $d = \infty$. Three objects were selected, a school-house gable, a church spire, and a house on the hill. The results were as follows:

TABLE 1.—Measurement of larger distances; base $b = 36.6$ cm.; $2R = 9.4$ cm.; $i = 45°$.

	$\Delta N \times 10^3$	$(\Delta N - \Delta N_0) \times 10^3$	d(cm.)	d (miles)	$\Delta N \times 10^3$	$(\Delta N - \Delta N_0) \times 10^3$	d cm.	d miles
Hill..	30.1 cm.	0.0 cm.	∞	∞	26.6 cm.	0.0 cm.	∞	∞
Spire.	31.8 cm.	1.7	72,000	0.44	28.4	1.8	68,000	0.42
Gable	....	...	....	....	31.9	5.3	23,000	0.14

The results for the spire obtained on different days and with different adjustments are as close as the limit of the micrometer (10^{-4} cm.) admit. All were in agreement with the data taken from a city map. With a larger b and a larger R, the accuracy would increase as the product of these quantities, so that the results are entirely satisfactory, notwithstanding the limited range. Measurements from the roof of the laboratory, which would have been very onerous, in view of the improvised apparatus, were therefore abandoned. I may add that by screening off the direct ray (K), the apparatus may be adjusted (if out of order) by putting the two images of the reflected (L) rays in coincidence. The fringes with daylight are perhaps not as intense as one would wish and too many are visible, differing in this respect from the case where an intense light and a collimator are used. In these directions further work is necessary. With ordinary plate and vertical fringes, the two images will usually be one above the other. Vertical coincidence in such a case is secured by aid of a fine vertical cross-hair in the ocular. Since all coincidences are subject to this, its precise trend is not of importance. Complete coincidence may, however, always be obtained with the use of compensators, as indicated in § 10.

8. Indirect observations.—These refer particularly to the actual use of the large angular displacement $\Delta\theta$ of the group of fringes in the ocular as

a whole, where $\Delta\theta = 2\Delta N \cos i/e$. If Δx corresponds to $\Delta\theta$ on an ocular micrometer and if the length of the telescope is L,

$$\Delta\theta = x/L \text{ or } \Delta x = 2L\Delta N \cos i/e$$

so that Δx may be very large as compared with ΔN. If, therefore, the distance of one object in the field is given, the distance of others might be advantageously found from $\Delta\theta$ or Δx. There is a difficulty, however, since $\Delta\theta$ and e vary together; i e., $d = bR/e\Delta\theta$. From this it follows that

$$d - d' = (dd'/bR)\ (e'\Delta\theta' - e\Delta\theta)$$

and only for such small differences of d and d' as may leave e appreciably constant could $d - d'$ be regarded as proportional to $\Delta\theta' - \Delta\theta$.

To get rid of the e it is necessary to introduce the fringe-breadth $\Delta\varphi$, which is measurable, while e is not. Since $e = \lambda/\Delta\varphi$ nearly, the equation becomes

$$d - d' = \frac{dd'\lambda}{bR}\left(\frac{\Delta\theta'}{\Delta\varphi'} - \frac{\Delta\theta}{\Delta\varphi}\right)$$

But this is virtually counting the number of fringes which pass between d and d'. If there are n fringes the equation may be written

$$d - d' = \frac{d^2}{bR}\, n\lambda$$

If $d' = \infty$, this becomes $d = bR/n\lambda = bR/(\Delta\theta/\Delta\varphi)\lambda$, where n is zero if $d = \infty$. There are but few fringes visible, however, and hence such an equation has but the specified limited application for distances close together, if fringes only are to be used. An example of the use of the latter equation follows, $\Delta\theta$ and $\Delta\varphi$ being estimated by an ocular micrometer. As before, if $b = 36.6$ cm.; $2R = 9.4$ cm.; $\lambda = 6 \times 10^{-5}$ cm., it was found that

$$\Delta\varphi = 0.010 \text{ cm.} \qquad \Delta\theta = 0.6 \text{ cm.}$$

Hence

$$d = \frac{36.6 \times 4.7}{(0.61/0.010) \times 6 \times 10^{-5}} = 5 \times 10^4 \text{ cm.}$$

This is too large as compared with $d = 4.3 \times 10^4$ cm. above, but no more so than the difficulty of measuring $\Delta\theta$ for a group of fringes $\Delta\varphi$ and the estimated λ imply. These 60 odd fringes as they passed by definite points of the ocular could have been counted here. In conclusion, measurement in terms of $\Delta\theta$ require a brightening of fringes and a diminution of the visible number. The number here was about 20 or more and they were visible during a displacement of $\Delta\theta = 100\Delta\varphi$, about. They were not quite equidistant, moreover, decreasing about 10 per cent in distance apart toward the ends of the group. A narrow strip of white paper illuminated by sunshine and visible in the field was the best background for their illumination. A few experiments were made on the relation of $\Delta\theta$, ΔN, and $\Delta\varphi = 0.01$ cm. on the ocular micrometer. The results were

$\Delta N \times 10^3$	$\Delta\theta \times 10^3$	$(\Delta N/\Delta\theta)\times 10^3$
2.5	620	4.0
2.2	500	4.4
2.4	570	4.2
3.6	850	4.2
3.9	870	4.5

The mean value $\Delta N/\Delta\theta = 0.00425$ agrees with the theoretical value $\lambda/2 \cos i = 6\times 10^{-5}/1.41 = 0.0043$ quite as well as the observations warrant.

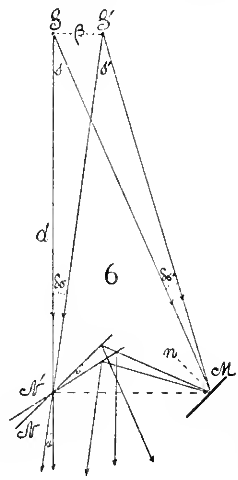
Another interesting application of measurement by fringes is shown in figure 6. This is concerned with the distance apart, β, of two objects S and S' (β parallel to b), both at a distance d from the observer. In passing from S to S' the angles at the base b are incremented by $\delta\sigma$ and decremented by $\delta\sigma'$, where

$$s - s' = \delta s + \delta s' = 2\delta s$$

if S is very distant relatively to b. But $2\ \delta\alpha = \delta s$; whence $\delta\alpha = \delta\sigma$ and $\beta = d\ .\ \delta s = d\ .\ \delta\alpha$. Since $2\ \Delta\alpha = 2\Delta N \cos i/R = n\lambda/R$, we obtain finally

$$\beta = d\frac{n\lambda}{2R}$$

if n fringes correspond to $\delta\alpha$, or the passage from S to S'.



Thus in case of the spire above, where $d = 7\times 10^4$ cm. or 0.43 mile, about 120 fringes were counted in passing from one side to the other of a conspicuous ledge of rock. Hence, since $2R = 9.4$. cm.,

$$\beta = 7\times 10^4 \frac{120\times 6\times 10^{-5}}{9.4} = 54 \text{ cm.}$$

9. Ellipses and hyperbolas.—The occurrences of spectrum ellipses and of hyperbolas, the rotation of fringes, centering of ellipses, etc., may be sufficiently explained as follows. Let the equation

$$(1) \qquad n\,\lambda = 2e\mu \cos r$$

refer to the horizontal axis, x, passing through the center of ellipses in the field of the ocular of the telescope. Here n is the order of a given dark fringe in wave-length λ, e the thickness, μ the index of refraction, and i and r the angles of incidence and refraction, all referring to the half-silver plate. There is no compensator. In the vertical direction in the spectrum the light is homogeneous in λ and the equation would be at the center upward or downward $n\lambda = 2e\,\mu \cos\beta$ . . . . . (2), if α and β are here the angles of incidence and of refraction (vertical plane). Since the angle α at least is very small, we may consider as an approximation that these equations hold throughout the field of the spectrum. Hence in general the e of equation (1) is to be replaced by $e \cos\beta$ in accordance with equation (1) and an

equation applying throughout the spectrum may therefore, under these simplifying conditions, be written

(3) $$n\lambda = 2e\mu \cos r \cos \beta$$

This changes to (1) or (2) according as β or r is zero. Moreover, if e and y are the linear coördinates of points of the spectrum and the length of the telescope tube is L, horizontal and vertical angles will be $\theta = x/L$, $\beta = y/L$, with the center of ellipses as an origin.

By the law of refraction (3) may then be changed to

(4) $$\mu\, n^2\lambda^2 = 4e^2\,(\mu^2 - \sin^2 i)\,(\mu^2 - \sin^2 \alpha)$$

Since α is small,

$$\lambda = D \sin \theta' = D \sin(\theta_0 + \theta) = D(\sin \theta_0 + \theta \cos \theta_0)$$

where θ is small in comparison with θ_0 (D being the grating space and $\theta' = \theta + \theta_0$ the angle of diffraction), we may write further

$$\mu\, n^2 D^2 (\sin \theta_0 + \theta \cos \theta_0)^2 + e^2\alpha^2 K = e^2 K \mu^2$$

where $K = 4\,(\mu^2 - \sin^2 i)$.

This is an ellipse if n at the center corresponds to a maximum value, in terms of the variables θ and α, so long as K and μ are considered constant. But as μ and therefore K vary with λ, though slowly, it is true the equation is more complicated.

When the center of ellipses is not in the field, but passes through the vertical plane corresponding to the center of the field of view, the ellipses may soon become appreciably straight lines in their *visible* contours, and the fringes must rotate in one direction or the other, according as the center is above or below the field. Rotation will be rapid when the vertical axis of the ellipses is relatively long. To bring the center into the field (for a proper value of N), the angle α must be zero, i. e., the two corresponding opaque mirrors which reflect the interfering beams must be rotated on a horizontal axis towards each other, or from each other, until $\alpha = 0$, or the horizontal plane through the field is a plane of symmetry.

Furthermore, since the fringes necessarily move toward or from the center of ellipses with change of N, the motion of fringes will necessarily be oblique if the center of ellipses is obliquely outside the field of view. In the limit, if the center is in the vertical plane specified, horizontal fringes will rise or fall.

Finally, if n passes through a minimum instead of a maximum, the fringes will be roughly of the hyperbolic type.

At the center of ellipses in case of spectrum fringes, n is therefore a maximum relatively to points of the spectrum in the same vertical or transverse line of homogeneous color. This maximum is due to obliquity, the horizontal one to change of λ. In the case of fringes produced with white light (without dispersion), like the colors of thin plates generally or the achromatic fringes discussed elsewhere, the center of ellipses (which are now circles) is an absolute maximum, horizontally or vertically, i. e., relative to points in all directions from the center and for each color. The center of

spectrum ellipses, therefore, has no direct relation to the center of white light fringes; for the latter occur only when the rays pass the plate normally. On the other hand, when the white-light fringes are straight lines corresponding to very oblique incidence of interfering rays, the spectrum fringes are none the less perfect ellipses.

It is finally necessary to account for the coincidence in adjustment of the center of spectrum fringes and the achromatic fringes, as the latter overlie the coincident white slit-images from which the superposed spectra are produced by the grating. This is easily seen to be referable to the fact that interferences with white light can only be visible if the light in the region of interference, when analyzed spectroscopically, contains but few dark bands. Since the number of bands in the spectrum is least near the center of ellipses, and is further reduced on making them as large as possible, the relation is obvious. In the case of strong, large achromatic fringes, a single fringe virtually occupies the whole spectrum. The light is either white or black.

The displacement of the center of ellipses with the angle of incidence for a given adjustment may be computed from the original equation for centers $N = e\,(\cos r + 2\,B/\lambda^2 \cos r)$, where r is the angle of refraction for the incidence i, e the thickness of plate, and B the dispersion constant. When N and e do not vary it may be shown that (since $\mu = A + B/\lambda^2$),

$$\frac{d\lambda}{di} = \frac{\lambda}{2B/\lambda^2}\,\frac{\sin r \cos i\,(2B/\lambda^2 - \mu \cos^2 r)}{\mu \cos^2 r\,(2 + \cos^2 r)}$$

To obtain an estimate $2B/\lambda^2 = 0.026$ may be neglected as compared with $\mu \cos^2 r$ and the equation given the approximate form ($\mu = 1.5$)

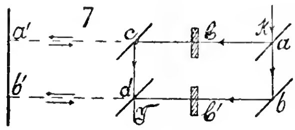
$$\frac{d\lambda}{di} = \frac{\lambda}{0.026}\,\frac{\sin i \cos i}{2.5\,\mu} = 10\,\lambda \sin i \cos i, \text{ nearly.}$$

Thus, if $i = 45°$, $d\lambda/di = 5\lambda$ or $0.09\,\lambda$ per degree of i, which is about 100 times the distance of the sodium lines. If $i = 0°$ or $90°$, the shift vanishes.

10. Compensators.—When the fringes are found they may be erected as stated by rotating either pair of the diagonal mirrors of the ray parallelogram towards or from each other, usually on a horizontal axis. The fringes may be enlarged by rotating the paired mirrors on either end of the ray parallelogram and restoring the fringes after each small step of rotation by displacement ΔN at the micrometer. But these processes are tedious and must be very cautiously performed or the fringes are liable to be lost. The same result may be accomplished by the aid of plate-glass compensators, about 0.5 to 1 cm. thick, placed normally in each of the two interfering beams and originally parallel and vertical. (See fig. 7, C and C'.) In addition to rotation and enlargement, these compensators serve with further advantage in equalizing the two beams in intensity. For this purpose it is merely necessary to half-silver lightly the compensator in the stronger beam of the ray parallelogram. If the fringes are more nearly vertical (between

45° and the vertical) they may be erected by rotating either compensator or both around a horizontal axis and enlarged by rotating around a vertical axis. The two compensators should be actuated together in opposite directions if there is danger of the fringes leaving the field; i. e., if the adjustment is considerable. Similarly, if the fringes are more nearly horizontal, and particularly when horizontal fringes are wanted, the fringes may be leveled by rotating the compensators in opposite directions around a vertical axis and enlarged by rotating around a horizontal axis. Horizontal fringes, which climb up and down the broad white slit-image and may be made quite large, are often very advantageous. The illuminated field is much more extensive in the vertical direction.

For slight adjustments it is convenient to have the compensators nearest at hand rotate in the same direction as the fringes. This may be done by working either on one side or the other of the center of fringes. The compensators may easily be manipulated by hand (without tangent screws) and they are most efficient when nearly normal to the respective beams. To pass from vertical to horizontal fringes one would first rotate the compensators in opposite directions around a horizontal axis until the fringes are inclined about 45°, after which the further rotations would be made in opposite directions around a vertical axis. The motion of fringes indicates the proper direction of the rotation of the plates.



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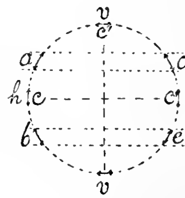

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To account for these apparently complicated effects, it is sufficient to recall that the compensators displace the center of fringes, usually enormously distant outside of the field of view, and besides that invisible with white light; for fringes are visible only in the narrow strip for which the spectrum fringes are very large and centered. Hence the result of rotation around a horizontal axis is to change fringes (fig. 8) of the type a, through c (vertical) into b, while the center moves downward, and vice versa. Again, rotation around a vertical axis changes fringes of the type a, through c' (horizontal) into d, as the center moves from left to right, and vice versa. The effects are necessarily opposite for the two beams. If the fringes are made vertical as at c, rotation of a compensator around the vertical axis can have no effect of rotation of fringes; for the center moves in the line of symmetry; but the effective or differential thickness of plates (e) is changed and hence the fringes are increased or decreased in size.

In view of the presence of compensators, C and C', figure 7, the original adjustment is much simplified; for it is necessary merely that the spots of sunlight on the mirrors at a and b, figure 7, and at a' and b', one or more meters off, be at the same level and at the same distance apart, nearly. The accurate adjustment at c and d for coincidence horizontal and vertical is then made with the telescope at T. When the distances are approximately equal,

fine spectrum fringes will nearly always appear. These are enlarged and centered as specified. A broad slit with the spectroscope removed will then show the achromatic fringes, which may in turn be enlarged and rectified. Horizontal fringes, though often convenient, are at fault, inasmuch as they must rotate with the displacement of the micrometer ΔN. This appears at once from figure 8, for the center of ellipses is shifted. Vertical fringes alone are free from rotation in relation to ΔN.

11. Number of fringes visible, etc.—To get the most promising conditions for observing coincidence in case of range finding, the direct and reflected images should be about equally intense. Hence, if α is the coefficient of reflection, the two equal intensities are

$$(1-\alpha^2)=2(1-\alpha)\alpha^2, \text{ or } \alpha=1/2$$

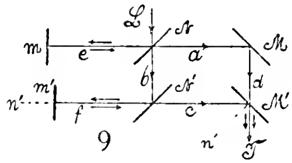
It is best, however, to have α less than this and to darken the direct beam if necessary by a thin half-silver plate interposed in the beam. If $\alpha=1/2$ the images are too dark and require higher illumination of the foreground than is usually present. As for the achromatic fringes themselves, they may be obtained with clear plate and opaque mirrors almost as well as with half-silvered plate, if the supernumerary images are partially screened off. With optic plate glass they would not appear. The surprising appearance of satellites—i. e., repetitions of the group with increasing faintness—is also common with clear plate.

A series of experiments were made by replacing the half-silver plates with grid-like opaque mirrors. These are easily made by removing the silver along parallel lines (using a T-square) with a sharp wet stick. The slit images were then also gridlike in appearance and the achromatic fringes occurred only on the dark bars. For clearly the superposition of beams takes place on reflection from glass only. In this way the fringes on the supernumerary slit images were identified. These occurred on the bright bars. The two phenomena are therefore complementary.

The reason for different numbers of visible fringes is less easily understood. In the original experiments two achromatic fringes (black or white), with about three green-reddish fringes on either side and rapidly fading out, were alone visible. This narrow grid is very advantageous if displacement interferometry is in question, for the achromatic fringes are easily recognized. Subsequently, however, large numbers of less distinctive fringes (20 or more) were usually obtained. As a small number of fringes is as frequently obtained with clear glass as with half-silvered plate, the occurrence is not attributable to the silver. (§ 13.) A variety of experiments were made with lenticular compensators, convex or concave, in each beam. The fringes, though obtained without difficulty, were usually rounded and irregular and the results without interest.

12. Separate adjustable auxiliary mirrors.—To obtain strong interferences the two component rays *eadT* and *bfcT* of the rectangular inter-

ferometer must not only be parallel on entering the telescope T (fig. 9), but they must be locally coincident at the mirror M; i. e., the two pencils from the collimator L (M',N,N' being half-silvers) finally entering T, must very nearly coincide. Otherwise, even when the path-difference is annulled and there is perfect coincidence of the slit image, no fringes may be obtained. This is often a great annoyance when the mirror M' is on a micrometer screw (n) normal to the face of the mirror; for on continuously displacing M' the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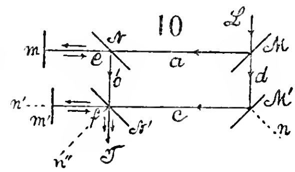
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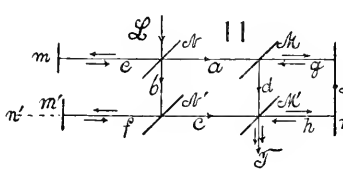
rays $cM'T$ and $dM'T$ separate more and more fully and the fringes soon vanish, unless a fresh adjustment for local coincidence is made. It is for this reason that the fringes are often so hard to find. The achromatics are much less sensitive to this disadjustment than the spectrum fringes; but the former are so mobile and easily lost that they have to be found as a rule by the aid of the latter.

To meet this difficulty there must be one mirror available which reflects the component beam normally and which may be displaced parallel to itself; i. e., whose micrometer screw is parallel to the incident and reflected ray. This condition is most easily secured by separating the auxiliary mirror into the parts m and m', each normal to its respective ray, while m' only is on a micrometer screw n'. Under these circumstances there is no difficulty in finding the fringes after the adjustment for parallel rays and local coincidence at M' has been made once for all by actuating the mirror m' in one direction or another. Moreover, it makes no difference, within limits, how the paral-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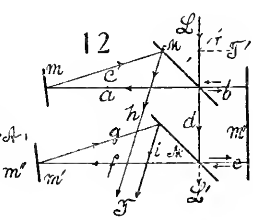

12

lelism and local coincidence are secured by moving any of the mirrors M,M', N,N', m,m', all of which must be on three leveling screws. Finally, if the mirror m and m' rotate as a rigid system about a common axis, it is still possible to use mm' for the measurement of small angles.

If the rays a and c, which may be of any length, are very long, the adjustment shown in figure 10 is preferable, as the observer at T is near the micrometer screw n'. Here M, N, N' are half-silvers.

For the case, however, in which the body whose small angles of rotation are to be measured is part of another apparatus which does not admit of manipulation, the method may be modified as in figure 11. Here all the plate mirrors M,M',N,N' are half-silvers and the rays from the collimator L form the interfering pencils $LemeagdT$ and $Lbfm'fct$. The mirror m and m' meet their rays normally and m' is in the micrometer screw at n' parallel to f. The mirror m'', on the axis A normal to the paper, rotates, and its small angles of displacement are to be measured. Considerable light is lost in the three penetrations of half-silver films by each ray; but in case of the achromatic fringes the light is usually in excess, so that the diminution of light is an advantage. It is more difficult, however, to find the spectrum fringes, as these require a slit.

The plan of figure 11 is carried out more simply in figure 12, where both reflections take place at the same mirrors M and M', respectively, the component rays being $Lbm''bachT$ and $Ldem''efgiT$. It is necessary to incline the parallel mirrors m' and m'' on a vertical axis, in order to avoid the entrance ray L' into the telescope at T. But this separates the component rays h and i locally, so that means must be employed (compensators, rotation of the other mirrors m'', N, or M') to obviate this as far as necessary. In both cases of figures 11 and 12 it is therefore not easy to find the fringes, and I did not persevere in the quest because of an eye affliction contracted at the time.

Similarly the system M, M', m'' of figure 12 might be used if a half-silver is placed at r and the telescope at T' to the right of it. In this case the mirror m'' must be in two parts, with adequate air-space inserted into the shorter ray Lb.

13. Types of achromatic fringes.—The difficulty of obtaining fringes of the strictly achromatic type (i. e., two strong fringes with a black line between and the remaining fringes green-reddish and faint) in the rectangular or other interferometer, has been frequently referred to in the text above. As a rule the fringes found are more or less diffuse, non-symmetric, with large numbers (10 or more) about equally strong. Such fringes are, of course, useless in displacement interferometry. When the sharp fringes needed are obtained, their definition is independent of the particular part of any of the glass plates used, and any plate may be rotated 180° in its own plane without spoiling the sharpness of the fringes. Hence such slight curvatures or wedge-shapes as the plates may possess are without influence on the phenomenon. To further test this I devised a screw-press adapted to push the vertical edges of a plate to the rear and the middle forward, so as to give the plate marked cylindricity. Quarter and eighth inch plates were operated on, in the latter case sufficiently to give the two superposed slit-images quite unequal width; but no essential or useful improvement of the fringes was observed. The type of the fringe was not altered. Again, the symmetrical fringes may be obtained from plates thickly or thinly silvered, without essential difference.

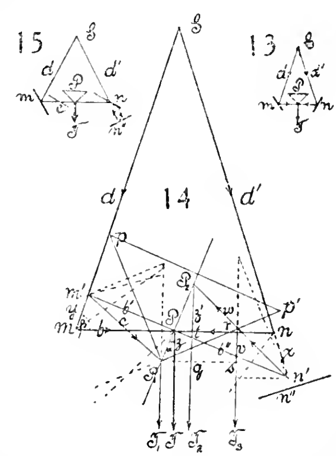
It follows, therefore, that the relative thickness of the glass paths traversed by the interfering beams can alone be of influence in shaping the fringe pattern in the manner in question. This is in consonance with the general theory of achromatic fringes, the result being a superposition of the color phenomena due to the dispersive refraction of the glass and the colors resulting from the wave-lengths of the interfering rays. To test this the apparatus, figure 10, is particularly convenient, as the fringes are easily found. Moreover, both rays, a and c, from the collimator at L, eventually pass through the plate N' before reaching the telescope at T. It is thus merely the thickness of the half-silvers M and N, both at 45°, that is here in question. If this thickness is the same, the sharp symmetrical design of but two strong fringes appears. If the difference of thickness is but little over 0.5 mm., many fringes, non-symmetric in distribution, are the rule. If the differential thickness is several millimeters there may be hundreds of fringes. If these are small they may be enlarged at pleasure; but they are always faint and useless for measurement.

CHAPTER II.

THE INTERFEROMETRY OF SMALL ANGLES, ETC.—METHODS BY DIRECT AND REVERSED SUPERPOSED SPECTRA.

14. Introductory.—It occurred to me that a number of the methods treated in my papers on direct and reversed spectrum interferometry might be used directly for the measurement of small angles and possibly of the distance of the source of light. Such a procedure would have an apparent advantage, at least theoretically, of not calling for the preliminary superposition of two images of distant objects, as the superposition is inherent in the method itself. But there are large constants involved, which make the result very problematical unless these constants can be removed by a compensator. Indeed, it is also very questionable whether such interference can at all occur. A further difficulty which hampers the method is the decrease of size of objects as their distance increases. Nevertheless a progressive investigation with the object of ascertaining to what degree the experiment is feasible is worth while, and as it will be convenient to develop the methods without reference to the ulterior conditions which limit the interferences, this method has been pursued.

15. Method with prism.—Figure 13 is a sketch of one of the methods in which S is the distant source of light, from which rays d and d' strike the mirrors m and n, and are thence reflected to the silvered sides of the right-angled prism P. After leaving it the rays enter the spectroscope at T in parallel. If the proper angles are selected the prism P may be replaced by one of any angle or by a reflecting grating.



Suppose now the system mPn is securely attached to a rigid metallic beam or rail capable of rotating around a vertical axis at its center (P). This is indicated in figure 14, where the direction of rays and the normals of mirrors have been drawn and where the angle of rotation α has placed mPn into the position $m'Pn'$. The result is that a part y of the ray d is cut off on the left side and a part x added to the ray d' on the right side, so that the path-difference, which may be assumed to have been zero originally, is now appreciably incremented, but not symmetrically for both sides.

It may be shown, however, that the rays $n'P_2T_2$ and $m'P_1T_1$ still enter the telescope in parallel and that therefore the conditions of interference have

not been disturbed. This is the interesting feature of the method, for the angle α between the two positions of the rigid beam will also be the angle between all corresponding normals of the mirrors, as indicated in the diagram. If we take the case on the left, the angle between incident and reflected ray at P will therefore be $\pi 2+4\alpha$ for the original mirror at P and $\pi/2+4\alpha-2\alpha=\pi/2+2\alpha$ for the new position at P_1. But the angle between the rays reflected at m and m' respectively as 2α. Hence if T_1P_1 is prolonged backwards it must intersect the line mn at the original angle $\frac{\pi}{2}$ and thus P_1T_1 is parallel to PT. Similar reasoning applies on the other side for P_2T_2 and will still hold if the direction of the ray Sn prolonged is *reversed*. Finally, $\pi/2$ may be any reasonable angle.

It will contribute to a more adaptable design of the apparatus for general interferometry if the ray Sn' may also be reversed by reflection (fig. 15, mirror n'') in parallel to itself, allowing a small lateral offset, similar on both sides for clearance of the mirrors. Reflection between fixed parallel mirrors on the left in d and between mirrors set at a reentrant right angle on the right, say at n'', would accomplish this at corresponding distances for the transverse rays. Again, half-silvers may be used at m and n for reflection, which method is probably best. These details will here be disregarded. If small angles are to be measured the direct method is enormously more sensitive.

16. Estimate.—The full expression for the path-difference corresponding to the rotation of rail α will be complicated and of no interest here. It is not sufficient to regard the intercepts y and x as solely contributing to the path-difference, which would therefore be $x+y$ for the direct case and $x-y$ for the case when the ray d' is reversed somewhere at n'' (fig. 15) and returned parallel to itself. It may be shown that for small angles α, if β is the angle between incident and reflected rays originally at m and n and b the distance $mP=Pn$, d the distance $Sm=Sn$,

$$n\lambda=2b\alpha-2b\alpha^2+2b^2\alpha/d \qquad\qquad n\lambda=2b\alpha^2-2b\alpha^2/d$$

are sufficiently approximate equations up to the squares of small quantities to meet the interference for the direct and the reversed cases respectively. Hence, if for instance $\alpha=1°=0.0175$; $b=1$ meter $=10^2$ cm.; $d=1$ kilom. $=10^5$ cm; $\lambda=6\times10^{-5}$ cm., the number of fringes corresponding to each of the terms may be computed as

$$\text{(Direct) } n=6\times10^4-10^3+60 \qquad\qquad \text{(Reversed) } n=10^3-60$$

In the first case over 61,000 fringes pass per degree of rotation, $\alpha=1°$. This makes about 2.9×10^{-7} or about 0.06 second of arc per fringe. But the method is insensitive as regards distances d, unless the first two terms can be compensated. In the second or reversed-ray case, the method would be relatively much more sensitive as regards d if the first term $2b\alpha^2$ could be compensated. The difficulty lies in the occurrence of α^2 in the term, whereas most compensators would act as the first power of α.

Furthermore, if the angle α is small and S is displaced over an angle α or a distance $r=d\alpha$ to the right, the original triangle may be regarded as restored. Hence the same number of fringes roughly should pass back again. In the second case, supposing $2b\alpha^2$ can be removed by compensation, $r=d\alpha$ and $\alpha=\lambda d/2b^2$, nearly, or

$$r=\lambda d^2/2b^2=6\times10^{-5}\times10^{10}/2\times10^4=30 \text{ cm.}$$

or the object should be located to 30 cm. at a kilometer for each fringe passing. In this case d need not be known, since

$$r=d\alpha=\frac{2b^2\alpha^2+b^2\alpha^3}{n\lambda}$$

and n fringes are observed to pass for the angle of rotation α in the compensated apparatus. The direct rays without compensation would of course give indefinitely better results if d is known; for the angle per fringe has been found as $\alpha=2.9\times10^{-7}$ when $r=d\alpha=0.03$ cm. per fringe if d is 1 kilometer.

Unfortunately, however, the method of figure 14 can not be rigorously carried out experimentally. For in any practical apparatus the mirrors M and N would have to rotate at a fixed distance from each other, apart from the micrometers; i. e., the two mirrors rotate on a rigid radius or rail and are therefore both rotated and displaced. It is this displacement which is relatively of much importance and by it all terms involving the first order of distance d are wiped out, so that terms of the second order in b/d only remain.

17. Equations.—To derive these equations certain intercepts of the rays, figure 14, in addition to x and y, b and b' may be defined. $P_1\,P_2$ is the trace of the vertical plane of symmetry of the right-angled prism, if rotated at an angle α to the right. In this case the reflected ray $n'P_2q$ on the right corresponds to the reflected ray $m'P_1$ on the left, both terminating in the common wave-front P_1qs before entering the telescope.

Let $n'P_2=c'=b\sin\beta/\cos\alpha\sin(\beta-\alpha)=z^1/\sin\alpha\cos\alpha$

$m'P_1=c\;=b\sin\beta/\cos\alpha\sin(\beta+\alpha)=z/\sin\alpha\cos\alpha$

$P_2t=z'=b\sin\alpha\sin\beta/\sin(\beta-\alpha)$

$tq=z\;=b\sin\alpha\sin\beta/\sin(\beta+\alpha)$

and $nn'=x=b\sin\alpha/\sin(\beta-\alpha)=z'/\sin\beta$

$mm'=y=b\sin\alpha/\sin(\beta+\alpha)=z/\sin\beta$

since the original angles at the ends of the base are β and the rotation α. The angles between incident and reflected rays are respectively $\beta-2\alpha$ at n', $\beta+2\alpha$ at m', $90°-2\alpha$ at P_2, and $90°+2\alpha$ at P_1. Most of the angles are indicated in the figure. The new radii $m'P=b'=b\sin\beta/\sin(\beta-\alpha)$; $n'P=b''=b\sin\beta/\sin(\beta+\alpha)$.

The rays, however, do not reach the planes of symmetry, but are reflected by the faces of the right-angled prism, and this may be sketched in, in the rotated position (angle α) at P_1pp'. The path of the reflected rays from n'

is now $n'rs$ and from m', $m'P_1$ before they meet in the common wave-front P_1qs. Hence the intercepts

$$rs = v = (z+z')\ (\cos\alpha - \sin\alpha)/(\sin\alpha + \cos\alpha)$$
$$rP_2 = w = (z+z')/\cos\alpha\ (\sin\alpha + \cos\alpha)$$

will enter in treating the path-differences. On the left the rays have not been distributed.

If we take the direct case first the original path-difference SnP and SmP may be regarded zero or n and m in the same phase. On rotation, therefore (angle α), the path-difference is increased on the right by $x+c'-w+v$ and increased on the left by $-y+c$, so that the total path-difference is equivalent to the equation

$$n\lambda = c' - c - (w-v) + x + y$$

If the above equivalents are inserted, this equation may be reduced to

$$n\lambda = 2b\sin\alpha\,\frac{\sin\beta\cos\beta/\cos\alpha - \sin^2\beta\sin\alpha + \sin\beta\cos\alpha}{\sin(\beta+\alpha)\sin(\beta-\alpha)}$$

in which the three terms in the numerator correspond to the respective intercepts $c'-c$, $w-v$, $x+y$.

Since α and β are small angles, we may write $\sin\alpha = \alpha$, $\cos\alpha = 1-\alpha^2/2$, and $\cos\beta = b/d$. Therefore the equation would, for practical purposes, become

$$n\lambda = 2b\alpha - 2b\alpha^2 + 2b^2\alpha/d$$

the three terms corresponding to the xy, wv, and cc' effect.

In the case of reversed ray (fig. 2) we may consider the points m' and n' in the same phase. Hence the original path-difference ($\alpha = 0$) is $x-y$. The path-difference after rotation $c'-w+v-c$. The total change of path-difference due to rotation is thus given by

$$n\lambda = c' - c - (w-v) - x + y$$

This differs from the preceding by the deduction of $2x$. The rays again terminate in the common wave-front P_1qs to enter the telescope. Hence after reduction

$$n\lambda = 2b\sin\alpha\,\frac{\sin\beta\cos\beta/\cos\alpha - \sin^2\beta\sin\alpha - \sin\alpha\cos\beta}{\sin(\beta+\alpha)\sin(\beta-\alpha)}$$

the terms showing the cc', wv, and xy effects. The approximate form of this equation is thus practically

$$n\lambda = 2b\alpha^2 + 2b^2\alpha/d$$

The wv effect predominates, the cc' effect is intermediate, and the xy effect very small if d is large, as already instanced.

The preceding equations may also be obtained geometrically by letting fall the normal from n (fig. 14) to the prism-mirror and prolonging the ray at s backward. In the isosceles triangle so formed the angle at the base is $45^\circ - \alpha$. Hence in the above notation the path-difference takes the form

$$x + 2\,(c'-w)\cos^2(45^\circ - \alpha) - (z'-z) - (c-y)$$

On inserting the values of the quantities as given above and reducing, an equation identical with the above appears, which for small α is

$$n\lambda = 2b\alpha\ (\operatorname{cosec}\beta - \alpha + \cot\beta)$$

If the prism has its nose at P (nearly), or in the axis of rotation, a small correction is to be added to the preceding expression. The path-difference on the right is increased by

$$z'\frac{1+\cos(90^\circ - 2\alpha)}{\cos\alpha\ (\cos\alpha + \sin\alpha)} + z'\frac{2\sin\alpha}{\cos\alpha - \sin\alpha}$$

and increased on the left by

$$z'/\ (\cos\alpha - \sin\alpha)\cos\alpha$$

Hence the correction is on reduction

$$2b\sin^2\alpha\sin\beta/\sin\delta = 2b\alpha^2, \text{ nearly.}$$

This merely wipes out the small middle term, $-\alpha$, of the above equation, leaving

$$n\lambda = (2b\alpha/\sin\beta)\ (1+\cos\beta)$$

When the prism is reduced to reflectors in its plane of symmetry, as treated at the beginning of this paragraph, the equation loses the terms $w-v$ and reduces to

$$n\lambda = x+y+z+z'+c'-c, \text{ or to } n\lambda = 2b\alpha\ (1/\sin\beta + 1)$$

In the practical apparatus the mirrors m and n rotate on a fixed radius b, whereas b in the diagram elongates on the right and contracts on the left respectively to

$$b'' = b\sin\beta/\sin\delta \qquad b' = b\sin\beta/\sin\sigma$$

Hence the mirrors in the apparatus are displaced normally on the right and left by

$$e = (b''-b)\cos\beta/2, \text{ inward, and } e' = (b-b')\cos\beta/2, \text{ outward.}$$

The path-difference thus introduced is the sum of the decrease on the right and increase on the left and its value is $2e\cos i$, when i is the angle of incidence in question. Thus the correction is (after reduction)

$$(b''-b)\ (\cos\alpha + \cos(\beta-\alpha)) + (b-b')\ (\cos\alpha + \cos(\beta+\alpha))$$

The expression may be further reduced to

$$\frac{2b\alpha\cos\beta\sin\beta}{\sin\sigma\sin\delta}(\cos\beta + \cos\alpha)$$

when α is a small angle, or to

$$2b\alpha\ (1/\sin\beta - \sin\beta + \cot\beta)$$

If this quantity is deducted from the above equation for path-difference and direct rays there remains simply $n\lambda = 2b\alpha\sin\beta$. The latter, therefore, is the equation to be used in interpreting the observation. So that generally when $i = \beta/2$ for the micrometer at n

$$2\cos i\ \Delta N/\Delta\alpha = 2b\sin\beta$$

In the case of reversed rays the conditions on the left remain the same as before. But on the right the mirror n is set at an angle $\beta/2$ to the rail

and at right angles to its former position. Hence the normal displacement is $e=(b''-b)\sin\beta/2$. The angle of incidence is $i=90°-(\beta/2-\alpha)$. Thus the path-difference here to be deducted is $2\,e\cos i$ or

$$2(b''-b)\sin\beta/2\ .\ \sin(\beta-\alpha)=(b''-b)\ \left(\cos\alpha-\cos(\beta-\alpha)\right)$$

and the total deduction from both sides is therefore

$$(b''-b)\ (\cos\alpha-\cos\delta)+2(b-b')\ (\cos\alpha+\cos\sigma)$$

This expression when reduced gives for small α

$$2b\alpha\ (\cot\beta-\alpha\cot\beta/\sin\beta)$$

or more simply

$$2b\alpha\cot\beta$$

It is practically as large as the total path-difference for reversed rays found above. If, therefore, the two effects are opposite in sign, the path-difference introduced by rotation would be zero, apart from the change of glass paths and second-order effects which are relatively small. In fact, the experiments show that the rotational effect, $\Delta\alpha$, in case of reversed rays, is relatively negligible as compared with the effect in case of rays not reversed. In other words, if from the equation for direct rays

$$n\lambda=2b\alpha\ (1/\sin\beta+\cot\beta)$$

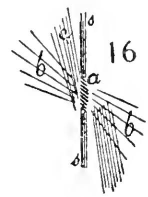
we deduct

$$2x+2e\cos i+2e'\cos i=2b\alpha\ (1/\sin\beta+\cot\beta)$$

the right-hand member vanishes to the second order of small quantities.

18. Observations. Prism=prism method.—In this case (fig. 19 below) a sharp-angled prism at S, with its knife-edge vertical, cleaves the beam of white light issuing from a collimator, reflecting the beams d and d' as described in my earlier papers. The system should be leveled so that all corresponding rays lie in a horizontal plane. By making the strips of light on both mirrors m and n (figs. 13, 14) coincide horizontally and vertically (using an auxiliary lens, if necessary) and then placing the prism P so that the rays mP and nP all but escape at its edge, the adjustment may be completed by aid of the telescope at T. The two slit-images, which should be equally bright, are made to coincide horizontally and vertically by the adjustment screws on m and n. If now the direct-vision spectroscope (prism grating) is swiveled in front of the objective of T, fringes will usually appear when the path-difference is annulled. For this purpose the prism P is placed on a Fraunhofer micrometer with the screw in the direction mn. The spectrum fringes are as a rule easily found and are quite strong, but they can not be centered in the field of view, for the occurrence of ellipses presupposes the rigorous superposition of the two strips of light on the edge of the prism P, which is not possible. The fringes, if too oblique, may be erected by a plate compensator with a horizontal axis, or the prism P may be rotated on a horizontal axis. Vertical spectrum fringes are not usually wanted in these experiments, for they are to serve only as an essential aid to finding the achromatic fringes.

It is a curious fact that although the ellipses can not be produced nor the slit much widened, apparently achromatic fringes do occur in white light for a micrometer placement at P such as should produce centered ellipses. Moreover, as the white slit-image is linear, the achromatic fringes are preferably made to run transverse to it. They are then exceedingly brilliant, extending much beyond the slit-image, and they travel up and down it with the motion of either micrometer at P or at n, (ΔN), or with the rotation of the rail $(\Delta\alpha)$. As there are but four or six fringes with but one or two strong and brilliant, they make an exceedingly sensitive index for measurement. The occurrence of achromatic fringes may also be detected in the solar spectrum, as all the Fraunhofer lines (homogeneous light) become helical and broad from the cross-hatching due to the fringes. Here with homogeneous light the fringes are indefinite in number and follow each other continuously, whereas with white light but one or two intense black-white fringes appear. Though the achromatic fringes are by far the most brilliant part of the phenomenon, they rarely occur without streamers. The general appearance is roughly suggested in figure 16, where ss is the white slit-image in the telescope and a the achromatic fringes moving up and down ss when ΔN or $\Delta\alpha$ change. In the lateral glare of the field, however, fan-shaped or radiating coarse fringes bb are seen, intersected with very fine hairlike fringes cc. Probably there is also an intermediate group. These streamers are very useful to register the approach of the achromatic fringes, which move so rapidly that they are easily lost.



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A few measurements or rather estimates were made to coördinate the values of ΔN of the micrometer displacement at n and the corresponding rotation $\Delta\alpha$ of the rail necessary to annul this displacement. To do this the achromatic fringes were placed on the cross-hair, or better, on the image of the cross-hair at the slit of the telescope, and both readings were taken. They were then displaced by rotating the rail and restored by moving the micrometer. To measure the rotation an index was placed at the end of the rail (radius 27 cm.) moving over a millimeter scale observed with a lens. The constants of the triangle, figure 13, were

$$b = 20 \text{ cm.} \qquad d = 62 \text{ cm.} \qquad \beta = 71.3^\circ \qquad i = 35.6^\circ$$

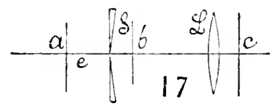
Corresponding readings were found as follows in two separate adjustments:

$10^4\Delta N$	$10^4\Delta\alpha$
0	0
204	9
493	19
Mean $\Delta N/\Delta\alpha = 26.5$ cm./radian	

$10^4\Delta N$	$10^4\Delta\alpha$
0.0	0
26.9	7
76.0	30
79.8	33
132.5	56
178.3	74
179.3	74
204.0	85
Mean $\Delta N/\Delta\alpha = 25$ cm./radian	

Since $\Delta N/\Delta\alpha = \dfrac{2b \sin\beta}{2 \cos i} = 23.3$ the observed data are above the computed values, but not more so than the difficulties of these measurements on an improvised apparatus imply. A much more refined method for finding $\Delta\alpha$ is, of course, essential.

19. Interference from rough surfaces.—The question now at issue is whether the interferences can be retained when the collimator is removed and the light comes directly from a ground-glass surface or a Nernst filament. The spectrum fringes go at once when the slit is widened; not so the achromatic sets. Having produced them clearly with sunlight, I found that a ground-glass screen or a scratched mica film could be placed at c or b or a, figure 17, whereas S is the slit and L the collimating lens. The fringes should be transverse, as in figure 16, as vertical fringes are too easily confounded with the white slit-image. The slit was now broadened or quite removed; but the fringes, though less prominent from excess of non-interfering light, remained in place distinctly and without other change. On removing the lens, however, the fringes invariably vanished.



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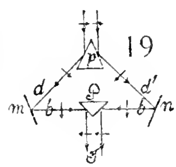
I now replaced the sunlight by the light of a Nernst filament, under the impression that ground glass might to a small degree still behave like plate glass. The same experiments were made, the filaments at e (fig. 17) replacing the ground glass. In this case, however, I first removed the lens L and it was then seen that the two washed slit-images were not superposed, as is otherwise obvious; but it accounts for the failures of the experiments with sunlight. Superposing the two vague images both out of focus, a position was soon found in which the achromatic fringes appeared brilliantly. The slit could now be widened or removed at pleasure, yet the fringes persisted strongly, but with loss of brilliancy.

It is thus possible to obtain these achromatic fringes directly from the Nernst filament and without a collimator; but they are so mobile, with change of $\Delta\alpha$ and ΔN, that to find them it is necessary first to produce the spectrum fringes with collimator and spectro-telescope; then to find the achromatic fringes on removing the spectroscope; next to remove the lens of the collimator and adjust for superposed images; and finally to remove the slit. These non-collimated achromatic fringes are best seen in a particular focal plane of the telescope and they change their focal plane with displacement ($\Delta\alpha$, ΔN). They practically cover the whole width of the washed slit-image. They usually measure about 0.5° in width, but the streamers may extend laterally five times further, depending on the adjustment. When pronounced, the slit-images may even be separated as in figure 18, while each alone retains the achromatic fringes. This puzzling phenomenon, which I had previously obtained, is probably due to the intersection and interference of rays in a region in advance of the plane of vision. Finally, as

the transverse arrow in figure 19 (indicating the right and left side of the slit-images) show, after the reflections at $p\,m\,n\,P$, although the slit-images are not reversed, the superposed rays are reversed; for these constitute the right-hand and left-hand radiation from the filament, separated by the prism p. If the slit is widened or removed, there is only one vertical line of rays (coinciding with the position of the slit before removal) which can interfere. The remaining light does not interfere and its admixture robs the phenomenon proper of its brilliancy.

18



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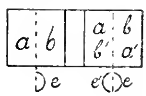
A few experiments made on the nature of the achromatic phenomenon here obtained showed that the fringes are probably Fresnellian interferences. To test this the objective of the collimator was removed and strong fringes were obtained by passing the two washed images of the slit over each other laterally, by moving the corresponding adjustment screw on the mirror n. It was found that the fringes passed from horizontal maxima in size, gradually to vertical hair-lines, as the images slid horizontally from contact of their nearer edges to the contact of the further edges. The coarse fringes were even strongly present in the narrow gap between slit-images before contact. The telescope was now focussed on the slit, so that sharp linear images appeared. The fringes vanished; but it appeared that the coarse fringes corresponded to coincident sharp slit-images when observed out of focus, and the fine fringes to sharp slit-images far apart. The whole phenomenon thus depends on the distance apart of two lines of light and the interferences are observable before or behind their plane.

20. Reversed rays.—The apparatus was now adjusted for the reversal of the rays d' by adjusting a mirror at some place n'' (fig. 15) on a fixed micrometer and in such a way that the rays on reflection retraced their path. The mirror at n being a half-silvered plate, in turn reflected the rays toward the prism P. This modification of apparatus introduces very considerable path-difference, $2\,nn''$, on the right, which must therefore be compensated on the left. It is difficult to accommodate the micrometer and leveling devices at n and n'' without an allowance of 5 to 10 cm. of path-excess. In my first experiments, which were merely tentative, the compensation on the left was secured by inserting a glass column about 15 cm. long. With this and the right-and-left micrometer displacement of the prism, or the to-and-fro motion of the mirror n'', path-difference was easily annulled and the ellipses found in the spectrum. They are centered as usual by rotating the glass compensator on a horizontal and a vertical axis, till with the occurrence of parallel rays at T the illuminated strips on the prism coincide, locally, to the eye.



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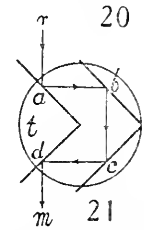

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In view of the long glass path and therefore of considerable dispersive effect,

the ellipses are small and the spectrum is filled with innumerable lines. Moreover, in view of the prism separation (at p, fig. 19) the ellipses are throughout half-ellipses, all terminating in the vertical axis. For the two areas or strips of light (ab, fig. 20) seen on the face of the grating and entering the spectro-telescope are each single, being one-half of the full area of light rays capable of interference obtained at the collimator. This results in the half-ellipses e. If the prism is replaced by a half-silver plate as in the next paragraph, the strips ab' and $a'b$ are both double, the full areas being superposed; thus the areas ab and $a'b'$ give rise to the full ellipses ee'. Hence, also, the vertical axis in e, being at the edge of the prism P, is not quite clear. Horizontal lines do not occur. These half-ellipses move with displacement of the micrometer at n', or at P, or on rotation of the rail mPn, as a body. It is difficult, however, to use them for measurement, as their vertical terminus is not sharp enough. If ΔN is the micrometer displacement corresponding to the rotation $\Delta\alpha$, we may write

$$2\Delta N/\Delta\alpha = 0$$

I did not succeed, however, in obtaining trustworthy results with the half-ellipses.

The achromatic phenomenon can not occur when glass columns are used for compensation from the great number of lines in the spectrum. To obtain large ellipses the dispersion effect B/λ^2 must practically vanish. Hence an air-path compensator is to replace the glass column. This is conveniently made (as shown in fig. 21) of two pairs of parallel opaque mirrors ab and cd. The pair ab are clamped between short lengths of square brass tubing and cd similarly and at right angles (nearly) to the pair ab. Both are mounted normally to a horizontal brass table t, provided with three leveling-screws, capable of being raised and lowered and of rotating around a vertical axis. The path-excess introduced is thus equivalent to ab and cd and the ray dm is collinear with ra. This compensator not only introduces path-difference, but since the mirrors are capable of rotating as a whole both around a vertical and a horizontal axis (leveling-screws), the beam dm may be moved right and left or up and down without ceasing to be parallel to ra. If, therefore, the ray entering T (fig. 19), were first made parallel, the ray d may be adjusted by the compensator until the strips of light on P practically coincide at its edge.

With the use of this air compensator or offset, the fringes were found without much difficulty and enlarged as specified. In view of the reflection at P, only half fields are returned; full ellipses or horizontal lines are not obtainable, as explained. But on removing the spectroscope and cautiously advancing the micrometer at N, the achromatic fringes eventually appear. In the present experiments these fringes did not take the usual and desirable form, consisting of but few fringes with the middle member in black and white. Probably because of the many reflections at mirrors (fig. 21), none of which was perfect, the fringes were now colored and present in large number

without much distinction between fringes. On being made transverse to the white image of the fine slit, they cross-hatched it from top to bottom. Nevertheless, their rapidity of motion is such that they serve quite well for measurement, the datum being more accurate than the measurement of $\Delta\alpha$. The comparison was carried out in the same manner as before, the presence of the achromatics being successively destroyed by rotation ($\Delta\alpha$) and restored by the normal displacement (ΔN) of the mirror at n. In this way the following data among many others were obtained. It is necessary to displace the mirrors very carefully; for if the fringes are lost they are extremely difficult to find without beginning with the spectrum fringes all over again.

$\Delta\alpha\times 10^3$	$\Delta N\times 10^3$	$\Delta\alpha\times 10^3$	$\Delta N\times 10^3$
0.	0.0 cm.	×14.4	16.0 cm.
2.6	2.1	18.2	19.0
6.7	7.0	21.8	21.4
8.2	9.4	25.5	25.9
11.5	12.3		

The range of $\Delta\alpha$ is much increased by removing the objective lens of the collimator, and this is done after the observation marked x in the table. The fringes are perhaps even more distinct when present in the absence of the lens. The constants of the apparatus were:

$$b=21 \text{ cm.};\quad \beta=70.7^\circ\ ;\quad d=64 \text{ cm.}$$

From this the rate $\Delta N/\Delta\alpha=1.05$ was found graphically. In the other series the rates were above 0.9. Approximate estimates of the same value were obtained with the spectrum ellipses and the glass column. This result again differs from the computed value, $\Delta N=0$. The reason may lie in the fact that the plane of symmetry of the prism P (fig. 14) did not pass through the axis of rotation, or was not originally midway between the mirrors m and n. To test this inference (which will again be treated in the next section) the following experiments were made:

The prism P was as carefully as possible centered by the eye, so that its plane of symmetry passed through the axis of rotation. In this case the relative measurements

$10^3\Delta N=0.0$	1.5	4.5	6.6	8.1 cm.
$10^3\Delta\alpha\ =0.0$	2.2	4.8	7.4	9.3

showed a mean coefficient of $\Delta N/\Delta\alpha=0.87$. Finally the prism was moved to the right, i. e., with its plane of symmetry on the other side of the axis. The results were

$10^3\Delta N=0.0$	3.0	4.4	7.0 cm.
$10^3\Delta\alpha\ =0.0$	3.0	4.4	6.7

giving a mean rate $\Delta N/\Delta\alpha=1.05$. Thus the shifting of the prism right and left has made but little difference and can not account for the discrepancy.

It is probable that the coefficients found are largely due to the half-silver glass mirror n (fig. 15), which rotates with the rail mn. To test this a com-

pensator c of the same thickness and glass may be placed in the beam mP on the left. If c and n are parallel, both originally at an angle $\beta/2$ to the beams traversing them, it is obvious that the compensation will not be destroyed by the rotation of the rail, provided c is fixed while n rotates. If c is effectively thicker than n, the part of the coefficient due to the compensator may become negative. This is apparently the case in the following experiments, in which a compensator was installed as in figure 15 at c:

$\Delta N \times 10^3 =$	-0.0	$-.4$	-1.4	-2.8	-4.0	-5.2 cm.
$\Delta\alpha \times 10^3 =$	0.0	2.6	5.9	9.3	12.2	16.0

The rate here is $\Delta N/\Delta\alpha = -0.31$, so that the zero value is exceeded. However, the path-difference in the compensator of thickness e at an angle of incidence i and refraction r, viz, e $(\cos i - \sin i \tan r)\ \alpha$, where $i = 90° - \beta/2$, here becomes $0.77\ (0.816 - 0.578 \times 0.618)\alpha = 0.353\alpha$, so that the whole difficulty is not explained away.

Finally, a few experiments were made to compare the effect of displacing (ΔN) the micrometer at n'' (fig. 15) as compared with the effect of a micrometer ($\Delta N'$) which displaces P in the direction of its plane of symmetry. The latter ($\Delta N'$) is zero when $\alpha = 0$. Generally if e is the normal displacement of the prismatic faces, the path-difference is

$$2e\ (\cos\ (45° - \alpha) - \cos\ (45° + \alpha)) = 2\Delta N \sin \alpha$$

since $e = \Delta N \sin 45°$. Hence, as ΔN is a normal displacement

$$\Delta N/\Delta N' \sin \alpha$$

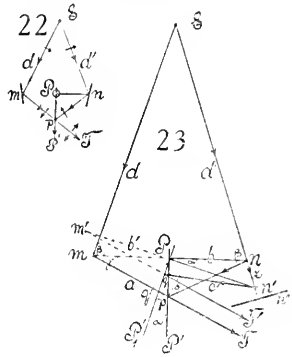
The results obtained were

$10^3\Delta N' = 0.0$	50	100	150	200	225 cm.
$10^3\Delta N\ = 0.0$	1.5	3.4	5.4	7.2	9.2 cm.

Thus the mean rate is $\alpha = \Delta N/\Delta N' = 0.036$ or α is a little over 2°. To obtain these data the achromatic fringes were used as above. When the slit-images seen in the telescope are not quite parallel, they may be made so by slightly rotating the slit on a horizontal axis normal to its plane. The images rotate in opposite directions. A slight angle between the images is, however, of no consequence.

21. Second method.—In view of certain difficulties encountered in the use of reflecting prisms, in particular the loss of rays at the edge, the limitation to half-ellipses, etc., the method of figure 22 enlarged in figure 23 was devised. In this the prism is replaced by a half-silvered plate PP'. Hence the rays issuing at S and reflected by the opaque mirrors at m and n are thereafter respectively transmitted and reflected by the half-silvered plate at p, and then reach the spectro-telescope at T together. When the path-differences are sufficiently equal, elliptic interference fringes will be seen in the spectrum. When first found they are usually very fine straight lines; but they may be rectified by plate compensators in the beams d and d' or mp and np, though

the operation is not easy. Leaving these details for further consideration, the procedure for angular measurement may advantageously be treated here. For this purpose the half-silver P and one opaque mirror, n, for instance, are mounted on a rigid bar with an axis at P. The other mirror m is to remain fixed. If the bar is now rotated over a small angle α, figure 23, the mirror at n is displaced to n' and the ray Sn prolonged (intercept x) is now reflected from n' to q and thence along T' into the spectro-telescope, parallel to its original direction or to the other ray mp. Hence the interferences remain intact, but many fringes pass during the transfer. The persistence of parallelism is easily seen, because the angle between the incident and reflected ray at n is decreased by 2α when n passes to n', but is again increased by 2α owing to the rotation of PP^1_1 to PP' over the angle α.



To control the fringes either the mirror at n (or at m) may be displaced on a micrometer screw normal to itself, or the half-silvered plate at P may be displaced parallel to itself. If the angle of incidence at n is i and the normal displacement of n is e, the path-difference introduced will be $2e \cos i$. Similarly if the normal displacement of the plate P is e' and the angle of incidence i', the path-difference will be $2e' \cos i'$.

As in the preceding experiment, the mirror at n may be a half-silver, so that the ray d' passes through it and may then be returned in its own path by a mirror at n'' on a fixed standard. The displacement of this mirror over a distance e, parallel to itself, introduces the path-difference $2e$, so that the cosines are avoided. But a much more important result is the fact that the rays np or $n'g$ now are stationary. The strips of light originally at p do not therefore travel over each other while one passes from p to q and the interferences are kept at full intensity throughout. This is a great advantage. Moreover, the half-silver plate at n compensates the half-silver Pp, which is a further advantage, since both paths within are glass paths with high dispersion coefficients. It is obvious that the path-excess nn'' on the d' side, must be separately compensated on the d side. The method of doing this by an air compensator (fig. 21) will presently be considered, as a long glass compensator would not in general be desirable because of the sluggish motion of the small ellipses thus produced.

22. Equations.—The equations for this case are apparently very complicated. If in figure 23, m and n are in the same phase and Pp is symmetrical, there will be no path-difference at p. When Pn is rotated over an angle α into Pn', the path on the right becomes $nn'+n'q+qs$ while (ps wave-front) the path on the left remains mp as before. The path-difference is thus the difference of these quantities, to which, however, the increased glass path at PP' would have to be deducted, and the surface PP', must pass through the axis P.

If the angle SnP is β and Pnp is γ, the values of the branch paths may be found to be (since $nP=mP=b$), if $\beta-\alpha=\delta$ and $\gamma-\alpha=\tau$,

$$np=mp=b/\cos\gamma$$
$$nn'=b\sin\alpha/\sin\delta$$
$$nq=b\sin\beta/\sin\delta\sin\tau$$
$$qs=\frac{b}{\sin\delta\cos\tau\cos\gamma}\left\{\begin{array}{l}\sin\alpha\sin\beta\sin\tau+\sin^2\gamma\sin\delta\cos\tau\\ \qquad -\sin\beta\sin\gamma\sin\tau\cos\tau\end{array}\right\}$$

Hence the path-difference is equivalent (after some reduction) to the equation

$$n\lambda=\frac{b}{\sin\delta\cos\tau\cos\gamma}\left\{\begin{array}{l}\sin\alpha(\cos\gamma\cos\tau+\sin\beta\sin\tau)+\sin\beta\cos\gamma\\ \qquad -\cos^2\gamma\cos\tau\sin\delta-\sin\beta\sin\gamma\sin\tau\cos\tau\end{array}\right\}$$

If $\alpha=0$, then $\beta=\delta$, $\gamma=\tau$, and the right-hand member becomes zero, as it should. I have not succeeded in putting this equation in a much more convenient form.

If α is very small, so that differential expressions may be introduced, the rigorous equation, to an approximation of the second order in α, may be reduced to

$$n\lambda=b\alpha\frac{\cos\gamma(1+\cos(\beta-\gamma))}{\sin\delta\cos\tau}=b\alpha\frac{1+\cos(\beta-\gamma)}{\sin\beta}$$

If β is nearly 90° and if the distance Pp is p, then

$$n\lambda=b\alpha\,(1+\sin\gamma)=b\alpha+p\alpha\cos\gamma$$

The same expression may be obtained geometrically by prolonging $n'p$ and $T'q$ to m'. The triangle $n'qm'$ is isosceles. Hence if we draw the wave-front Pq', normal to pT, we may deduct the common length pq' from both rays, or add it to both. Hence the path on the right will be $x+b'\cos(\gamma-\alpha)+p\sin\gamma$, where b' is the line Pn' and on the left as before, $b/\cos\gamma$. Hence (if $b=p/\tan\gamma$) the path-difference is

$$b\left(\frac{\sin\alpha}{\sin\delta}+\frac{b'}{b}\cos(\gamma-\alpha)+\tan\gamma\sin\gamma-\frac{1}{\cos\gamma}\right)$$

which, as above, also reduces to

$$b\alpha\,(1+\cos(\beta-\gamma))/\sin\beta$$

There is a source of discrepancy which enters when the face PP' does not pass through the axis P, but is eccentric. In such a case, if e is the distance of the plate from the axis, a correction equivalent to $e\,(1-\cos\alpha)\cos\gamma=e\alpha^2\gamma$, nearly, will have to be supplied. Again, if there is lack of symmetry from such a cause, the base-lines will be b and b' and the angles γ and γ', so that a modified equation is suggested. Finally, from all these expressions the changes of glass path on the left with α, if not compensated, must be deducted. As the method admits of a good achromatic phenomenon of reversed slit-images, it is theoretically interesting and I have given it some study.

For the case where the ray Sn prolonged returns on itself, as from n'' in fig-

ure 23, the mirror n being a half-silvered plate, the quantity $nn''=2b\alpha/\sin\varphi$ must be deducted. This makes the first term negative, so that the equation for reversed rays is, apart from sign,

$$n\lambda = b\alpha\,\frac{1-\cos(\beta-\alpha)}{\sin\varphi}$$

One may notice that when $\gamma=0$ these expressions coincide with the equations for the prism-prism methods above, except for the factor 2.

It is now necessary to apply the correction for the occurrence of a constant radius of rotation, whereby the mirror n is both rotated and displaced. This correction is, for a normal displacement e and an angle of incidence i, $2\,e\cos i$. If the distances $Pn=b$ and $Pn'=b''$

$$e=(b''-b)\sin(90^\circ-(\beta-\gamma)/2)=(b''-b)\cos(\beta-\gamma)/2$$

The angle of incidence is now $(\beta+\gamma)/2-\alpha$, so that the path-difference in question becomes

$$2(b''-b)\cos(\beta-\gamma)/2\,.\cos\left((\beta+\gamma)/2-\alpha\right)=(b''-b)\left(\cos(\beta-\alpha)+\cos(\gamma-\alpha)\right)$$

Since $b''=b\sin\beta/\sin(\beta-\alpha)$ if α is a small angle, this may be written $(b''-b)=\alpha b\cos\beta/\sin\delta$. Hence the correction is

$$b\alpha\,\frac{\cos^2\beta+\cos\beta\cos\gamma}{\sin\beta}$$

Subtracting this from the first equivalent of $n\lambda$, and reducing, the equation to be used for non-reversed rays becomes $\lambda n=b\alpha(\sin\beta+\sin\gamma)$ when α is small. Except for the factor 2 this result is again identical with the above for two prisms, if $\gamma=0$. As the angle of incidence at the micrometer at n is $i=(\beta+\gamma)/2$, and if ΔN is the displacement

$$2\Delta N\cos(\beta+\gamma)/2=b\,(\sin\beta+\sin\gamma)\,\Delta\alpha$$

For reversed rays, $e=(b''-b)\sin(\beta-\gamma)/2$ at once, and the path-difference is then for small α

$$2b\alpha\cot\beta\sin\frac{\beta-\gamma}{2}\cos\left(90^\circ-\frac{\beta+\gamma}{2}+\alpha\right)=b\alpha\cot\beta\left(\cos(\gamma-\alpha)-\cos(\beta-\alpha)\right)$$

since the angle of incidence is now $90^\circ-(\beta+\gamma)/2+\alpha$. Subtracting this path-difference from the above equivalent of $n\lambda$, the practical equation for reversed rays becomes, after reducing,

$$n\lambda=b\alpha\,(\sin\beta-\sin\gamma)$$

and since $i=0$, then

$$2\Delta N=b(\sin\beta-\sin\gamma)\Delta\alpha$$

Both equations contain the distance d of a remote object in $\sin\beta$.

23. Observations.—In the first experiments a sharp-angled prism was placed at S, reflecting the two rays d and d' of white light from a collimator beyond. The method of figures 22 and 23 with a micrometer at n was first used, the screw being in the direction of the bisectrix of the angle $\beta+\gamma$. The collimator is convenient, as otherwise the coincident slit-images are liable to separate

when the micrometer mirror moves. Moreover, quite a narrow silt is preferable, so that all the light reflected by the prism is that diffracted by the slit and issuing in parallel rays from the objective. Any further light escapes interference and blurs the fringes. The adjustment is not easy, as has been shown in the earlier papers. It is necessary that the two rays *mp* and *np* pass through the same vertical at *p* and additionally enter the telescope at *T* in parallel. When this is the case and the path-difference is annulled, the fringes are very black on the yellowish background of the spectrum near the *D* lines. In proportion as the rays separate into *pT* and *qT*, the fringes become fainter and finally vanish. Near *p*, however, they may be seen until they vanish at either elongation from smallness. Another reason for this marked tendency to vanish is the fact that the slit-images are mirror-images of each other. Hence if the slit-images are widened, even when the strips of light seen at *p* coincide, there can only be a narrow vertical region of actual coincidence of light of like origin. It follows, moreover, that the achromatic fringes proper can not be produced with white light by this method, though it is possible to produce the linear phenomenon with white light when the ellipses are vertically centered. The experiment, which is at first very difficult, will presently be treated. Movable fringes are sometimes seen on the white slit-images.

When first obtained the spectrum fringes are usually very fine parallel lines. To bring the center of ellipses into the center of the field a plate-glass compensator, capable of rotation on a horizontal axis, should be placed either in the ray *d* or *b*, or on the other side, as the case may be. When the compensator is set at the proper angle very dense black ellipses may be brought into the center of the field by the micrometer screw or by the rotation of the system *Pn*, figure 23. In fact, on using both of these displacements in a contrary sense (i. e., annulling the effect of slight displacements of the micrometer by a corresponding counter-displacement of rotation), the two beams may be made to pass through *p* together and without path-difference. In this case the ellipses are very strong. If both the mirrors at *n* and *P* are displaced contrariwise but parallel to themselves this may also be done. There is no such difficulty with the method of reversed rays.

The following measurements were made as a rough test of the equations given above. The constants of the apparatus were (fig. 4)

$d = 62.3$ cm. $\quad p = Pp = 10.6$ cm. $\quad \beta = 71.3^\circ \quad \gamma = 27.1^\circ \quad b = 20$ cm.

the prism angle therefore being about 18.7° and the divergence of rays from it 37.4°. A much *sharper* prism than this would have been desirable, so the *d* could have been larger, but none was at hand. The angle of incidence here is $(\beta+\gamma)/2 = 49.2^\circ$. Hence

$$\frac{2\Delta N \times 0.653}{\Delta\alpha} = 20\ (0.947 + 0.455) = 28.$$

The procedure for the test of the equation consisted in establishing the ellipses with the micrometer at *n* (reading *N*), rotating the rail over a small

angle (reading α), and re-establishing the ellipses again at the micrometer (N'). To find α, an index was attached to the end of the rail (radius of rotation 23.6 cm.) and the motion of this index over a glass millimeter scale was read off with a lens. For really refined work it would have been necessary to adjust a micrometer tangent screw to find α; but this did not seem called for here. The results were in case of two series ($2 \cos i = 1.306$):

ΔN	$2\Delta N \cos i$	$\Delta\alpha$	ΔN	$2\Delta N \cos i$	$\Delta\alpha$
0.0 cm.	0.0 cm.	0.0	0.0 cm.	0.0 cm.	0.0
0.1357	0.1777	0.0076	0.0607	0.0793	0.0034
0.0577	0.2531	0.0101	0.0541	0.1530	0.0068
0.0462	0.3135	0.0139	0.0664	0.2397	0.0098
0.0718	0.4073	0.0160			
Mean $\frac{2\Delta N \cos i}{\Delta\alpha} = 24.5$			Mean $\frac{2\Delta N \cos i}{\Delta\alpha} = 24.0$		

These coefficients (which are here below the computed value) were found graphically. The discrepancy for the correction due to increasing obliquity of the plate Pp is but (e thickness, μ index of refraction, r angle of refraction, the incident being $i=\gamma$)

$$e\,(\sin\gamma - \cos\gamma \tan r)\Delta\alpha = 0.14$$

per unit of α, if $e = 0.775$ cm., $\mu = 1.55$, and $\gamma = 27.1°$.

A further discrepancy may be sought for in the fact that if the surface Pp (fig. 23) does not pass through the axis of rotation, this plate is both rotated and displaced. The error so introduced may be either positive or negative. If the displacement of plate is $e' = e\,(1 - \cos\,\alpha)$, the equation should read

$$\frac{2\Delta N \cos \pm i 2e' \cos\gamma}{\Delta\alpha} = b\,(\sin\beta + \sin\gamma)$$

where e is the distance of the plate from the axis. The following experiments were therefore made with some consideration to greater symmetry of apparatus. The constants were: $\beta = 70.5°$, $\gamma = 27.1°$ and therefore $i = 48.8°$, or $2 \cos i = 1.32$. The data found were in different adjustments;

$\Delta N \times 10^3$	$\Delta\alpha \times 10^3$	$\Delta N / \Delta\alpha$	$2\Delta N \cos i / \Delta\alpha$
110.9 203.7 246.5	5.2 9.6 11.6	21.2	28.0
102.9 208.4	4.8 10.0	20.5	27.1
47.6 80.1 170.4 223.6 257.6	2.6 4.1 8.5 10.7 12.7	20.7	27.3

These results agree much better with the theoretical equation than the former, and may be considered as coinciding with it.

In the present case the attempt to get interference from rough surfaces was not at first successful. The slit-images are reversed, as indicated by the transverse arrows in figure 22. Hence if the white slit-images are wide there can be coincidence only in a single vertical line. Fringes with white light will occur as a case of the interference of fine slit-images. To produce them it is first necessary to obtain the spectrum fringes with the ellipses, or else with horizontal fringes in the field. If now the spectroscope is removed and the white slit-images put out of focus, the phenomenon indicated in figure 24, where *s* is the superposed, washed slit-images will usually appear, or may be found on cautiously moving the micrometer screw. Within the slit-image the fringes are coarse and colored, but they send out fine oblique streamers into the field of diffuse light or glare, on both sides of *s*. When the slit is widened these fringes are liable to vanish just as the spectrum fringes vanish, except perhaps at the edges of the images. These achromatic fringes climb up and down the slit-image with motion of the micrometers (ΔN, $\Delta\alpha$) with extreme rapidity and are easily lost, as there are not usually more than 10 or 20 of them. If the spectrum ellipses are huge, the white fringes are almost too coarse to be seen and too mobile to be controlled.

24


25


I next removed the objective of the collimator. The fringes, though much changed in appearance, practically black and white, were not destroyed. In such a case the slit-image shrinks vertically. To obtain a long strip a highly illuminated ground-glass screen (sunlight and weak condenser) should be placed in front of the slit as a source of very diffuse light. In such a case this long white post (as it were) is covered from top to bottom with sharp blackish and usually oblique lines, which vanish at once, up or down, on moving the micrometer. No fringes are seen if the slit is in focus. When considerably out of focus (as in case of the diffraction patch in fig. 25,) strong, sharp-colored cross-markings are present, which would be quite available for measurement. However, in this experiment, when the slit was widened or removed, the fringes apparently vanished. The phenomena as a whole seem to me to be fringes of the two white slit-images, and seen either behind or in front of their focal plane, like the complementary fringes described elsewhere. This is confirmed by experiments presently to be described.

24. Reversed rays.—The apparatus was now adjusted for a reversal of rays by putting a half-silver plate at n and an opaque mirror on a micrometer (with the screw normal to its face) at some position n'' (fig. 23) fixed independently of the rotation. In this case, therefore, the intercept nn' changes sign. Moreover, the angle of incidence at n'' is 0°. Hence the equation should be

$$2\Delta N/\Delta\alpha = b\ (\sin\beta - \sin\gamma)$$

The constants of the apparatus were

$$b=20 \text{ cm.} \qquad \beta=71.3^\circ \qquad \gamma=34.9^\circ$$

Thus

$$2\Delta N/\Delta\alpha=20(0.947-0.572)=7.5$$

To obtain the ellipses a thick plate-glass compensator may be placed in the d ray to provide for the elongation $2x$ in d'. About 14 cm. of glass column were necessary. This makes it very easy to center the ellipses and to obtain them intensely black on a colored ground by rotating the compensator on a horizontal and vertical axis until the two strips of illumination at p quite coincide when the rays T, T' are parallel. But on the other hand, because of the thickness of glass used, the small ellipses obtained move relatively sluggishly with displacement of the micrometers. The sensitiveness decreases proportionately to the thickness of glass path.

Experiments, of which the following data are examples, were made by alternately restoring the center of ellipses to the D lines of the solar spectrum first by the micrometer (ΔN) and thereafter by the rotation of rail ($\Delta\alpha$). The adjustments were very different.

$\Delta N \times 10^3$	$\Delta\alpha \times 10^3$	$\Delta N \times 10^3$	$\Delta\alpha \times 10^3$
0.0 cm.	0.0 rad.	0.0 cm.	0.0 rad.
53.2	13.7	24.3	1.9
66.1	17.0	33.3	4.4
58.1	14.8	——	——
44.1	11.5	43.3	7.4
21.1	5.2	57.4	10.7
		68.8	14.0
Mean $2\Delta N/\Delta\alpha=7.6$		Mean $2\Delta N/\Delta\alpha=6.8$	

The second series changed its rate enormously, almost one-half, owing to necessary intermediate adjustments (inserting a new zero). Otherwise the observations are as good as the apparatus permitted; but the computed $2\Delta N/\Delta\alpha=7.5$ is above the observed value, usually, possibly owing to an eccentric position of the plate Pp relative to the axis of rotation. To test this point of view the plate Pp was displaced eccentrically toward the left of the axis. The result should be a modified coefficient, but the following data obtained in the same way as before fail to bear this out, however:

$$\Delta N \times 10^3=0.0 \qquad 19.9 \qquad 40.2 \qquad 56.8 \text{ cm.}$$
$$\Delta\alpha \times 10^3=0.0 \qquad 5.9 \qquad 12.2 \qquad 16.7 \text{ rad.}$$

The rate, $2\Delta N/\Delta\alpha=6.6$, does not differ essentially from the above. With these small ellipses there can not, of course, be an achromatic phenomenon. To obtain large ellipses the glass-path difference (i. e., the dispersion) must be abolished on both sides and an air-path difference introduced, preferably in a way which has been shown above in figure 21. As such experiments are so much more trustworthy and sensitive, I did not pursue the glass-column work further.

25. Fringes from rough surfaces.—Experiments were now made with the use of the air-path compensator (fig. 21,) placed in the *d* rays when these rays were reversed. A magnificent set of achromatic fringes were here found, only about five in number, with the central members in black and white. Tests similar to the above showed that they are Fresnellian interferences. To prove this the objective of the collimator was removed and even more brilliant fringes were found on placing the washed slit-images in contact. If these patches of light were slid over each other horizontally, by moving the adjustment screw for rotating the micrometer mirror on a vertical axis, the fringes rotated nearly 180°, passing from vertical hair-lines, through a maximum of coarseness for the horizontal fringes, back to hair-lines again. On focussing the telescope on the slit it was then found that the large horizontal fringes corresponded to coincident slit-images in focus, whereas for the very fine fringes the focussed slit-images are far apart. No fringes appear on the slit-images in focus, in any case. They lie in front of and behind the image plane. This is exactly the case found above, except that here the edges of the washed slit-images are exchanged.

The endeavor to obtain the fringes without the slit was next tried. For this purpose a ground-glass screen illuminated by sunlight (*a*, fig. 17) was first placed in front of the slit *S* in the absence of the objective (*L*) of the collimator. The fringes were still very prominent, though the light was darker. The slit *S* was now also removed. The fringes could then no longer be seen; but on narrowing down the illuminated ground-glass screen *a* to a vertical strip of light 1 to 2 mm. broad, they were unquestionably present. In such experiments, therefore, the chief function of the slit *S* is to cut off the light which does not interfere, so that the fringes are lost in the glare. In the absence of such excess of light the fringes are quite visible and therefore certainly always present. By aid of the offset air compensator huge achromatic fringes may be easily produced; but they are so sensitive as not to be manageable in an improvised apparatus.

A number of measurements were now made with the achromatic fringes set at convenient (small) size by the air-compensator. In this work the plate PP' (fig. 23) was moved in three steps over about 1 cm. For each step a set of data was investigated. The results were in succession

$$(1)\left\{\begin{array}{llllll}\Delta N\times 10^3=0.0 & 1.9 & 6.1 & 12.4 & 17.8 & 21.9\text{ cm.}\\ \Delta\alpha\times 10^3=0.0 & .4 & 1.2 & 2.3 & 3.3 & 3.9\text{ radian}\end{array}\right\}\frac{2\Delta N}{\Delta\alpha}=10.4$$

$$(2)\left\{\begin{array}{llllll}\Delta N\times 10^3=0.0 & 7.1 & 12.2 & 19.2 & 26.6 & 32.6\text{ cm.}\\ \Delta\alpha\times 10^3=.0.0 & 1.1 & 2.2 & 3.7 & 5.2 & 6.3\text{ radian}\end{array}\right\}\frac{2\Delta N}{\Delta\alpha}=9.6$$

$$(3)\left\{\begin{array}{lllll}\Delta N\times 10^3=0.0 & 9.3 & 19.2 & 27.7 & 37.7\text{ cm.}\\ \Delta\alpha\times 10^3=0.0 & 1.8 & 3.7 & 5.9 & 7.4\text{ radian}\end{array}\right\}\frac{2\Delta N}{\Delta\alpha}=10.2$$

The coefficients so obtained are practically identical; and they agree as nearly

as may be expected with the equation, since the angle B and γ are not easily specified with accuracy. They were

$$\beta=71.3^\circ \qquad \gamma=28.4^\circ \qquad b=21 \text{ cm.}$$

so that theoretically

$$2\Delta N/\Delta\alpha=21(0.947-0.476)=9.9$$

26. Direct interferences without cleavage prism.—The next step in advance was made by dispensing with the sharp prisms heretofore used for cleaving the rays issuing from a collimator (or the slit simply) in the endeavor to obtain two rays capable of interference. The assembly of apparatus is shown in figure 26, where S is the slit (to be replaced by a Nernst filament or a tungsten filament), m and n the opaque mirrors, pp' the half-silvered plate. The rays dd', diffracted at S, pass after reflection into c and c' and may be observed by spectro-telescopes placed either at T or T'. In the first experiments the distance Sp' was about 4 meters and the distance mn 10 cm. The mirrors n and pp' were on micrometers with the screws normal to their respective faces. The distance mn must be within the limits of the wedge of light from the slit and is therefore small, unless d is very large. Both pp' and n are on the rotating rail (as above), whereas m is fixed. The apparatus was also adjustable for reversed rays by attaching an auxiliary mirror, normal to the rays d' prolonged through n. S being distant, this slit must be long, as otherwise the spectrum band will be a mere horizontal line and the fringes difficult to detect. A doublet of lenses, each about 10 cm. in diameter and of the same focal power (1.60 cm.), but respectively convex and concave and having a combined focal distance of about 5 or 6 meters, is of advantage for focussing a large solar image, 1 to 2 inches in diameter, on the slit. The Nernst or tungsten filament gives the same advantages at once; but the former is too thick, at least for the initial experiments at shorter distances.

The fringes are exceedingly difficult to find in spite of the brilliant spectra. It was not until after about three days of searching, in which (besides sunlight) the filaments as well as the methods of direct and of reversed rays were used, that the experiment ultimately succeeded with sunlight. The filaments are much less gracious. To obtain the fringes calls not only for very accurate adjustment for horizontal and vertical spectrum coincidence, but the fringes lie quite sharply in a definite focal plane, usually between that of the slit-image and the principal focal plane; the rays must interpenetrate at the plate and finally path-difference must be nearly annulled. And there are other conditions presently to be stated. After being found they are quite strong elliptic spectrum fringes, but when lost nevertheless difficult to rediscover. The slit may be broadened till the spectrum lines vanish, perhaps to more than a millimeter, before they disappear in a uniform spectrum band.

The achromatics which coincide in adjustment with horizontal spectrum

fringes and are seen with the slit-image out of focus are also difficult to find because of the short length of the slit-image. As first obtained they lacked brilliancy and were not easily observed. Similarly, experiments with filaments failed to show the fringes, although made in parallel with the successful result with sunlight.

A considerable assistance in finding the fringes is an opaque screen with a vertical slit 2 mm. to 4 mm. wide placed just in front of the objective of the spectro-telescope, in the best position as to symmetry. This screen cuts out rays which do not interfere and makes the fringes stronger, even though the background is darker. Fringes are frequently found in this way when they are all but invisible in the full spectrum.

In addition to the regular fringes, a much larger vague set seems to be present in another focal plane. They also rotate, etc., like the regular fringes, but the experiment led to no decision with regard to them, as they appeared in the field erratically and could not be produced at will. They may be shadow interferences of the principal set.

An attempt was made to register slight lateral displacements of the slit in terms of the displacement of fringes, but as the slit-images are thrown out of coincidence when the slit moves, trustworthy numerical data can not be obtained. One may estimate that as a first approximation

$$x=\frac{d}{c}\lambda$$

if x is the lateral displacement of the slit and c the distance between mirrors m and n. Hence

$$x_0=\frac{400}{10}6\times10^{-5}=0.0024 \text{ cm.}$$

should have been equivalent to the passage of one fringe in the given apparatus, or generally

$$2\Delta N\cos(\beta+\gamma)/2=cx/d$$

If $(\beta+\gamma)/2=70°$, $\Delta N=10^{-4}$ cm., and $d/c=40$, then

$$x=2\times10^{-4}\times40\times0.34=2.7\times10^{-3} \text{ cm.}$$

could have been registered. Incidentally it appears that two vertical lines of the slit, $x_0/2=0.0014$ cm. apart, would wipe out each other's interferences; but this is not the case, as much greater slit-widths are admissible. To the right and left of the line of the slit capable of producing interferences the parallel lines either cease to produce parallel rays, or parallel rays come from symmetrical but different lines.

After completing these experiments, the distance between slit S and the mirrors m and n was increased to about 9 meters. The same lens doublet, focussing a large solar image on the slit, was used as before. With the aid of the slotted screen in front of the telescope and the micrometer distances from the preceding experiment, the fringes were found without difficulty.

In fact, in view of the longer distance *d*, the slit could be opened to over a millimeter of breadth before the fringes quite vanished from the spectrum; but on using a somewhat stronger condensing system (concave lens of doublet preceding the convex lens) and consequently more oblique rays, a very fine slit was needed to show the fringes. They are thus more easily found when the rays are more nearly parallel.

With artificial light, again, I obtained no results, even after long searching.

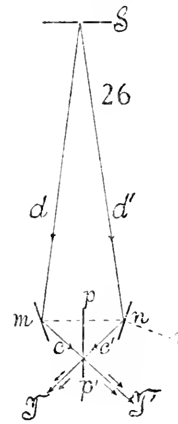
Operating with two successive slits at about 9 meters from the interferometer, one of which received the light through the other, I found that two independent sets of fringes very different in size and inclination could be put in the field together. The further investigation eventually showed that the size and inclination of the fringes is essentially dependent on the degree of parallelism of the two slit-images. When the images are parallel, the fringes are of maximum size and vertical. When the images are not quite parallel (they incline in opposite directions when the slit is slightly rotated in its own plane from the vertical), the fringes rapidly grow smaller and rotate. With parallel slit-images the spectrum ellipses are centered in the field; otherwise they are very far out of center. The adjustment for actual (not X-like, coincidence must therefore be made with precision if large fringes are wanted.

Further work was also done with sunlight to obtain more pronounced achromatics. For this purpose a compensator was inserted to equalize the glass path in the half-silvered plate. Huge spectrum ellipses were obtained in this way and their centers were placed above the telescopic field, so that the fringes seen were large horizontal bars. On removing the spectroscope and placing the slit-images out of focus, brilliant achromatics were in fact obtained, of the concentric hyperbolic type, vividly colored and broad between the apices, and diminishing to hair-lines laterally. With these it was possible to enlarge the slit to at least 3 mm., without destroying the fringes, though they became more vague. It is necessary that the slit-images, when in focus, should be quite parallel, otherwise any broadening of the slit will wipe out the achromatics. It was possible to place a plate of ground glass on the far side of the slit without destroying the fringes, but not on the side towards the interferometer. In other respects the behavior was as described in the case of achromatics in the earlier experiments with a cleavage prism.

Finally, the spectrum fringes and the corresponding achromatics were obtained with the light of a Nernst filament, at first by focussing an image of it with a strong condenser lens on the slit. The experiments, however, are very difficult. The spectrum fringes are often weak, out of focus, and extremely sensitive to small disadjustments in the horizontal and vertical coincidence of the slit-images. They require a fine slit. When well produced the achromatics are also obtainable on removing the spectroscope when the spectrum fringes are horizontal bars. The achromatics may also be obtained brilliantly without the condenser lens, but the adjustment must in such a case be made first with sunlight, as the spectrum from the Nernst filament

is too feeble for detecting fringes so elusive as the present. The achromatics, however, are strong and brilliant even here (Nernst filament).

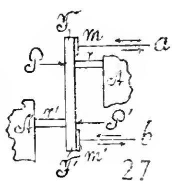
An interesting result is obtained in case of the achromatic fringes by narrowing one of the beams, for instance that coming from the mirror *m* (fig. 26), by a screen with a vertical slit about 2 mm. wide. In such a case the slit-image (out of focus) is correspondingly narrowed. It may be passed from side to side of the broad washed slit-image coming from the mirror *n*, by moving its adjustment screws (vertical axis). The fringes then appear only in a particular position of the narrow image in the field of the broader; but when they do appear they spread far beyond the margins of the narrow image on both sides. Interference thus apparently occurs where but one beam is present. The phenomenon is like those instances above (figs. 18, 24, Chapter II) and means, as I understand it, that the beams have met in some other focal plane, though one is tempted to conclude that interference is stimulated by resonance, in particular as it is often impossible to find a plane in which they have met. The achromatics may sometimes be seen before and behind the principal focal plane, but more frequently either in the one or in the other region only.



CHAPTER III.

THE ELASTICS OF SMALL BODIES.

27. Introductory method.—At the request of Professor W. G. Cady, who was in need of Young's modulus in case of certain crystals used in experiments in which he is interested, the endeavor was made to adapt the above interferometer for measuring small angles with an auxiliary mirror for this purpose. The project seems feasible and apparently simple in execution when the method of end-thrust indicated in figure 27 is used. Here F is a rigid metallic bar subjected to the force couple P,P', carrying the coplanar mirrors m,m' and capable of rotating slightly in a horizontal plane. The mirrors m and m' receive the corresponding rays a,b of the interferometer. The couple P,P' is resisted by the resilience of the rods r,r' to be tested, as these push respectively against the ends of the bar F and against the rigid abutments A and A' of the apparatus. If the couple P,P' changes, the bar F rotates correspondingly and the component rays a,b of the interferometer will register the amount of rotation by the methods given in Chapter I. Thus, if E is the traction modulus, l the elongation of each rod of length L and right section A under the force P,



27

$$E=\frac{\Delta P/A}{\Delta e/L}$$

Δ being a differential symbol. If the distance apart of the rays a,b is $2R$ and that of the force couple is $2R'$,

$$2\ R\Delta\alpha=\Delta N\ \cos i$$

if the bar F rotates over an angle $\Delta\alpha$, in consequence of the increment ΔP of thrust, and if ΔN is the displacement of micrometer mirror (angle of incidence is i), needed to restore the interference fringes to their original position. But

$$\Delta l=R'\Delta\alpha=R'\ \frac{\Delta N\cos i}{2R}\ \text{nearly};$$

so that after reduction

$$E=\frac{2LR}{AR'\cos i}\,\frac{\Delta P}{\Delta N}$$

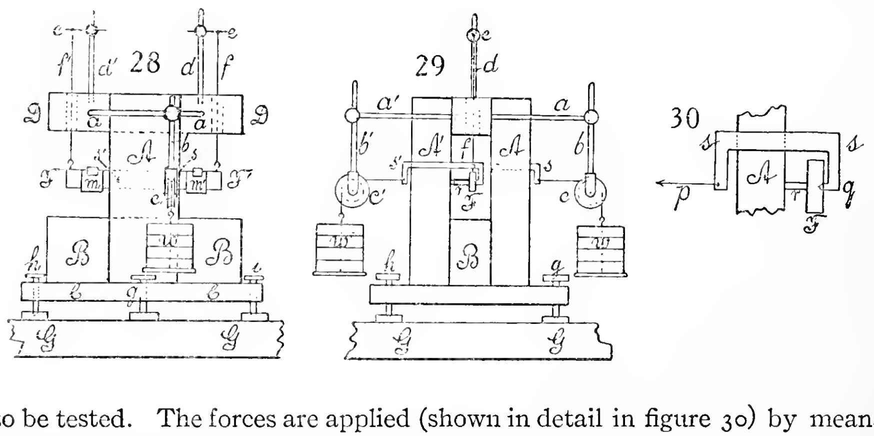
Since $i=45°$, $$E=2\sqrt{2}\ (L/A)(R/R')(\Delta P/\Delta N)$$

The method will not of course be very exact, because for rods less than an inch long the quantities involved, particularly ΔN, are so small. Any flexures or slight dislocations of parts of the apparatus are of relatively great

consequence. But there is a much more serious consideration. In long rods the stresses distribute themselves equally throughout the sectional area; but in short rods this is probably not the case. There will be lines of longitudinal stress and part of the area A will be relatively unstressed. Hence the values of E will come out too small and the question is rather to what degree such a method can be made trustworthy. With the optical method there will be no difficulty, if the achromatic fringes are used. The observed displacement of these is adequate if a reasonably thin rod is used and fringes need not be counted. It is not even necessary to make the method very sensitive. Fringes of moderate size suffice.

Should the method of end-thrust fail, the method of flexure is more liable to be successful, since the measurement of the sag at the middle of two rods placed at r and r' parallel to FF offers less serious difficulties. But such flexure involves continuously decreasing stresses from the middle to the ends of each rod and is thus fundamentally less interesting. For very short rods, moreover, theoretical difficulties similar to the preceding would be encountered.

28. Apparatus.—The apparatus used is shown in front (normal to the interferometer rays) and side elevation in figures 28 and 29. F is the bar carrying the mirrors m,m', already mentioned, and compressing the rods r,r'



to be tested. The forces are applied (shown in detail in figure 30) by means of a set of weights ww', the pulleys cc' and the rigid brass offset ss acting on the cone at q, which fits into a depression in F. The lines pq pass through the axes of the rods r or r'. In this way any flexure of the bar F is obviated.

The rigid part (abutments of the system) is made up of the cast-iron bricks A, A', B bolted together firmly. These rest upon the plate C provided with three leveling-screws g,h,i, g,i being parallel to the bar F. These leveling-screws are of special importance in controlling the size and inclination of the achromatic interference fringes, gi being the horizontal axis. The whole apparatus may be rotated around a vertical axis on the horizontal base GG, which is itself movable on slides nearer or further from the interferometer.

Size of fringes is controlled by rotation around the vertical. The iron bricks A,A',B were each $10\times2\times3.5$ cubic inches in my apparatus, certainly rigid enough for the purpose. The inner faces should be smooth and parallel. Between them at the top the wooden bar DD has been bolted in place which carries the rods of the railing aa', etc., with clamps for the support of the forks bb' of the pulleys cc'. It also carries the standards dd' and arms ee', to which two silk threads ff' are adjustably attached for the bifilar support of the bar FF', when not in use.

The stresses on the system thus seem balanced throughout, the ultimate tendency being a compression of A and A' in a horizontal direction, which is of course negilgible. The only discrepancy which may be effective is the possible flexure of the plate C by the varying weights ww'. It is for this reason that the weight of A,A',B was made excessive as compared with ww.

29. Preliminary observations.—The first experiments were made with hard-rubber tubes each $L=2.2$ cm. long and $A=1.46$ cm. in sectional area. The offset ss (fig. 30) was not at first applied and the beam F showed definite flexure, inasmuch as the white slit-images separated. With this tube of hard rubber a value of the order of $E=6\times10^9$ was obtained. With the offset (used throughout the following work) the apparent modulus increased to $E=9\times10^9$ and the white slit-images remained in contact. Thus far horizontal spectrum fringes had been the criterion of measurement. It was found just as easy and more accurate to use the achromatic fringes.

In three series with the same tube the values came out as

$$10^{-9}E=7.8,\quad 7.8,\quad 8.1$$

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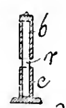


32

33

All these data are apparently much too low. Consequently the hard-rubber tubes a were provided with thick brass caps b, as shown in figure 31, the small conical projection fitting a re-entrant cone in the beam F. Nevertheless the value now found ($E=6\times10^9$) was even lower. There is thus no doubt that the section of the tube is not uniformly strained in a short solid, whereas in a long slender body the stresses are soon equalized. Part of the sectional area of the short solid may in fact be quite free from strain. Hence the device shown in figure 32 was next tested, where a relatively thin rod of hard rubber r is surrounded by brass caps b and c with closed ends. The caps fit the rod loosely and room is left between them for compression. For hard-rubber rods of the dimensions $L=1.7$ cm., $A=0.24$ cm.^{2}, and in three series of experiments the moduli $10^9E=6.9$, 7.5, 7.5 were computed, showing therefore no marked change from the preceding data, in spite of the greatly diminished sections. In the next experiments shaped rods like figure 33 were used directly. For the dimensions $L=2.2$ cm., $A=0.41$ cm.^{2}, the moduli were found to be

$$E\times10^9=9.6,\quad 9.6,\quad 8.4,\quad 8.4$$

in four series made after different periods of loading.

In all this work the initial loads of 1 kg. each were not removed. The elongations between 1 kg. and 4 kg. were consistent, though naturally with definite evidence of apparent hysteresis. For instance, on the last series with brass caps the individual contractions were

$\Delta P=$	4	3	2	1	2	3	4	kg.
$10^4\Delta N=$	120	94	55	0	40	78	141	cm.

and in the last series with the mushroom-shaped solid,

$\Delta P=$	4	3	2	1	2	3	4	kg.
$10^4\Delta N=$	89	68	38	0	34	64	89	cm.

As the values ΔN are proportional to the decrements of length, the rod is shorter on unloading and longer on loading, *cæt. par.* In other words, the hard-rubber rod is influenced by its immediately antecedent history. The curved lines obtained may, however, also be influenced by gradual dislocation or decreased firmness in the seating of the rod. Thus the modulus in the last example decrease from about $10^9E=9$ between the two highest loads (3 to 4 kg.) to $10^9E=6$ between the two lowest loads (1 to 2 kg.). Of the two the former (high loads) is unquestionably the more trustworthy and least influenced by imperfections of apparatus.

Results of the same nature were obtained for brass, though here, from the much greater rigidity, the effect of dislocation at small loads is much more apparent. The first experiments were made with a relatively thick rod, of the dimension $L=2.2$ cm., $A=0.70$ cm.^{2}, for which $E=4.5\times10^{10}$ was obtained. This is enormously too low and the result is a mere indication of the yield of the apparatus or inequalities of stress. The rod was now turned down on the lathe to a mushroom-shaped solid (fig. 33), the dimensions $L=2.1$ cm., $A=0.113$ cm.^{2}. With this the values $E\times10^{11}=1, 2, 5$, were found, according as the rod was unloaded or loaded from 4 to 3 kilograms. Clearly this is a case of dislocation of parts of the adjustment. The rod was then further diminished to diameter of but 0.2 cm., so that $L=1.8$ cm. and $A=0.035$ cm.^{2} now obtained. With this, for the highest loads, values $E=1.4\times10^{11}$ followed under like conditions. This is no improvement and shows that the apparatus yields seriously for moduli as large as that of brass (say 10^{12}). Thus in the series with largest values of E the individual contractions were

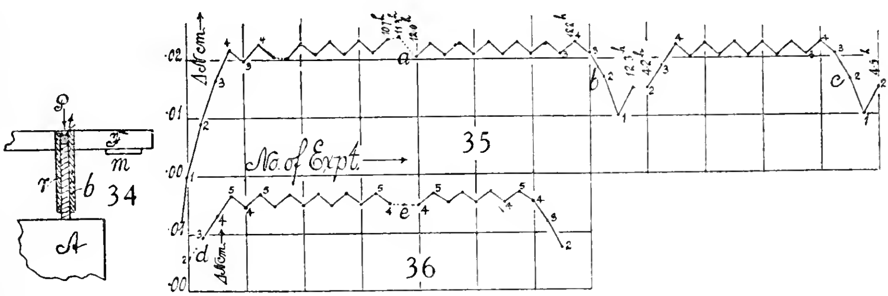
$\Delta P=$	4	3	2	1	2	3	4	kg.
$10^4\Delta N=$	1.6	1.4	0.9	0	.9	1.5	2.1	

30. Rods in metallic sheath.—In the data thus far obtained the value of E is throughout too low, showing that the section of the rod has not been uniformly stressed. A final modification was therefore made as shown in figure 34, in which the rod r (to be tested) is rather loosely surrounded by a rigid metallic tube or sheath rigidly screwed into the bar FF of the inter-

ferometer. This tube is closed at t with a tightly-fitting screw-plug provided with a conical depression to receive the thrust P from the point q of the offset ss (fig. 30). The rod r is thus compressed between q and the rigid abutment A of the apparatus, and projects but slightly (1 or 2 mm.) beyond the tube b. Special sheaths b are provided, fitting neither too tightly nor too loosely, for different diameters of rod r; or a number of coaxial tubes, b, may be telescoped for the purpose.

This device gave more satisfactory results at once, and with such bodies as hard rubber showed the change of modulus with stress, the occurrence of hysteresis, and viscous deformation. It is particularly interesting, inasmuch as it gives the apparent value of the modulus under each of these conditions.

The successive contractions (ΔN) and modulus values for hard rubber as found in several successsive series are exhibited in table 2 and figures 35 and 36. In the first series the relatively large micrometer displacements, ΔN,



are probably due to crushing or to fitting the unstressed rod to the abutments of the apparatus under increasing pressure. Thereafter these large contractions do not again occur. The moduli obtained from triplets of observation between 3 and 4 kg. gradually increase to a fixed value. In the second series the rod which had been loaded for some time (see table) with about 40 kg. per centimeter shows the limiting value of moduli found, $E=4.4\times10^{10}$. We may contrast with this the small modulus when the load is but 1 to 2 kg.

In the third series, beginning with a rod but slightly stressed, a low value of the modulus at first appears, but it soon reaches the limiting values again. In figures 35 and 36 the contractions (the numerals show the loads) are given in succession. With the exception of the necessary break at the beginning of the second series, the work is continuous without modification of adjustment. As contractions are positive the rod is gradually becoming shorter and more viscous. The data are throughout consistent, much more so than was expected. For instance, the effects of the removal of weights at b and c are practically identical. The triplets in figure 35 all show an upward slope or continuous viscous contraction of the rod under large end-thrust. In series

TABLE 2.—Modulus of hard rubber. End-thrust apparatus with sheathed rods; $L=2.45$ cm.; $2r=0.367$ cm.; $A=0.106$ cm.²; $i=45°$; $2R=10.3$ cm.; $2R'=7.0$ cm. Permanent load 1 kg. each. Achromatic fringes.

P	$\Delta N \times 10^4$	$E \times 10^{-10}$	P	$\Delta N \times 10^4$	$E \times 10^{-10}$	P	$\Delta N \times 10^4$	$E \times 10^{-10}$
kg.	*cm.*		*kg.*	*cm.*		*kg.*	*cm.*	
10^h30^m 0.5	−50		12^h0^m **3	105		4^h10^m 2	49	
1.0	0		4	128	4.41	3	87	
2	90		3	108		4	124	3.32
3	164		4	129		3	103	
4	219	2.53	3	108	4.41	4	126	
3	198		4	130		3	105	4.47
(1) 4	228		(2) 3	108		(3) 4	127	
3	206	3.70	4	130	4.47	3	105	
*3	204		3	109		4	127	4.37
4	230	4.18	4	129		3	105	
3	209		3	109		4	127	
4	233	4.13	4	129		3	106	4.43
3	210		3	109		4	128	
4	234		2	116		3	106	
3	211	4.13	1	0	1.67	2	62	
10^h45^m 4	235		12^h16^m 2	48		1	0	1.77
					(b)	4^h30^m 2	46	(c)
11^h20^m 4	240	(a)						
†2	47		5	167		***4	149	
3	85		4	148	5.12	5	169	5.34
4	120	3.41	5	167		4	153	
3	99		4	149		5	170	
			5	168	5.20	(7) 4	153	5.43
‡2	58	(d)	4	150		5	171	
3	94		5	168		4	153	
(6) 4	131		4	150	5.25	5	170	5.81
5	165	3.63	5	168		4	153	
4	146		4	151		3	120	
						2	73	

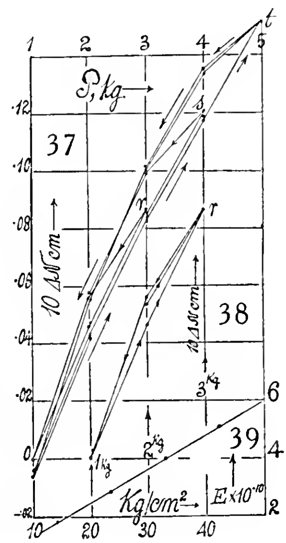
* Slight adjustment. ** After adjustment for larger fringes. *** 2 hours later.
† Next day. ‡ Two days later.

6 and 7 a special collection of data was investigated two days later at the highest loads which the apparatus admitted, 4 to 5 kg. These are shown after *d* (fig. 35) and with a further lapse of time after *e*. It is interesting to note that whereas the contractions under 5 kg. are relatively stationary, the contractions at 4 kg. increase in the successive loadings and hence E also increases. Otherwise the behavior, allowing for the fact that the rod had been stressed for several days, is about the same, *cæt. par.*, as before.

In figures 37 and 38 results[1] with special reference to hysteresis are shown as a whole, indicating the three successive loops terminating at *r*, *s*, *t*, for loads between 1 and 3, 1 and 4, and 1 and 5 kg. on the 0.106 sq. cm. of section. It is difficult to interpret the individual data, because clearly their nature is

[1] As the data are sufficiently reproduced by the curves, the numerical tables have been removed.

exceedingly complex. There may be some instrumental error; but it is interesting to note that in figure 37 the rod subjected to variations of loads between 1 and 4 kg. shows gradual expansion at the lower pressures. Just as the loops are evidences of hysteresis, so the gradual upward trend of successive loops between the same end-loads is evidence of viscosity.



Finally I collected the values of the modulus E, obtained under different loads P in the successive series of experiments. In each series the stable value was reached between given alternating end-pressures, gradually, as already explained. The mean values are

Loads =	14.2	23.6	33.0	42.4 kg/cm².
$10^{-10}E$ =	1.81	2.82	4.00	5.11

results which are exhibited in figure 39. They happen to lie on a straight line. Although these mean values are unequally influenced by the character and number of the experiments made, the result as a whole disarms suspicion. It is improbable, in other words, that data which have shown such detailed consistency as appears in figures 35 to 39 should be seriously influenced by imperfections of apparatus or of method, though it is possible that at the higher pressures the sides of the hard-rubber rods may have expanded into and been sustained by the walls of the rigid sheath b (fig. 34). In the above estimates the rate R at which the modulus E increases per kg./cm.² of pressure would be, nearly,

$$R = 10^{10}\frac{5.1 - 1.8}{42.4 - 14.2} = 0.12 \times 10^{10}$$

The rate R is excessive as compared with subsequent values.

31. Same. Thinner rods, hard rubber.—The suspicion left in the preceding experiments that the marked increase of E was possibly due to the lateral expansion of the hard-rubber rod, so that it more or less filled the rigid sheath at the highest loads, induced me to repeat the work with slightly thinner rods. The former case would make E approach the bulk modulus. The rods last used were therefore turned down on the lathe until their dimensions were

$$2L = 4.90 \text{ cm.} \qquad 2r = 0.35 \text{ cm.} \qquad A = 0.098 \text{ cm.}^2$$

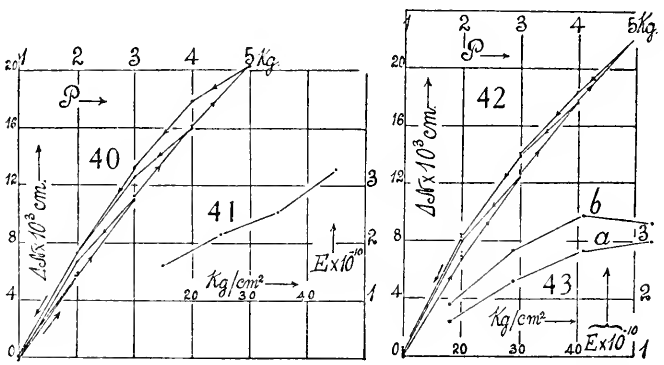
They were then tested on the interferometer for cyclically increasing and decreasing loads P, as shown in table 3, where ΔN is the micrometer equivalent of elongation. A few extra triplets were added. The results are also given in figure 40 and show a beautiful case of hysteresis with gradually increasing loops. This hysteresis is in no way less accentuated when compared with the preceding cases.

TABLE 3.—Hard rubber. Thinner rod. $2L=4.90$ cm.; $2r=0.353$ cm.; $A=0.098$ cm.^{2}; other data as above. Load kg. permanent. $R/R'=10.3/7.0=1.47$.

P	$\Delta N\times10^5$	$E\times10^{-10}$	P	$\Delta N\times10^5$	$E\times10^{-10}$	P	$\Delta N\times10^5$	$E\times10^{-10}$
kg.	*cm.*		*kg.*	*cm.*		*kg.*	*cm.*	
1	0		2	725		3	1150	
2	565		1	5	1.60	4	1660	
3	1103	2.15	2	585		5	2070	3.06*
2	670		3	1135		4	1800	
1	−20	1.62	4	1620		5	2225	
2	575		5	2035	3.18*	4	2590	3.76**
3	1125		4	1795		5	2200	
4	1605	2.52	3	1335				
3	1260		2	730		4	2460	
			1	15		3	2920	2.53
			1	0	1.58	4	2555	
			2	605				

* Loads 4, 5, 4 kg. ** Loads 5, 4, 5 kg.

The values of the modulus E were then computed from the triplets in succession. One may note that the succession 5, 4, 5 kg. gives a much larger value of E than the contrasting succession of loads 4, 5, 4, kg. One can not, there-



fore, expect a smooth march of values unless this difference is systematically included, while it would be exceedingly difficult to even conjecture a rational method of quantitative interpretation. With this conceded, the mean values were

$$\text{Load} = \quad 15 \quad 25 \quad 35 \quad 45 \text{ kg./cm.}^2$$
$$10^{-10}E = 1.60 \quad 2.15 \quad 2.53 \quad 3.32$$

These data are shown in figure 41. From them the mean rate $R=\Delta E/\Delta(P/A)=0.053$ may be deduced. This is in fact less than one-half the preceding ratio R, so that the suspicion that the rigid sheath containing the rod loosely contributes to the high E values at high loads is not removed. It is necessary to reduce the thickness of rods further.

In figures 42 and 43 cyclic data are given for the same rod turned down further to the following dimensions:

$$2L=4.90 \text{ cm.} \qquad 2r=0.33 \text{ cm.} \qquad A=0.085 \text{ cm.}^2$$

The cycles remain, though they are more slender than before. The general trend of the observations, indicating a continual slow viscous yielding throughout, superposed on the hysteresis, is the same as before. The change of the modulus with the load, i. e.,

$P/A=$	18	29	41	53	kg./cm.^{2}
$10^{-10}E=$	1.64	2.30	2.83	3.00	

is in the first three cases not very different from the preceding. The fourth datum is low. One may write $R=\Delta E/\Delta(P/A)=10^{10}\times 0.05$. It is questionable, therefore, whether lateral support received from the rigid sheath can account for the increment of E with the load. The variability of R is rather due to the occurrence of hysteresis, whereby the datum for E is very variable unless found in triplet observations between two definite loads. A few additional triplets (3 or 4 for each step of loads, added to elucidate this question) follow:

$P=$	1 to 2	2 to 3	3 to 4	4 to 5	kg.
$P/A=$	18	29	41	53	kg./cm.^{2}
Mean $10^{10}E=$	1.88	2.84	3.44	3.32	

These data are all larger than the preceding set, owing to the different method of observation (triplets confined to two fixed pressures and not to pressures varying cyclically over a large interval); but the general trend of results (see fig. 43*b*) is the same as before. As E is a maximum in the interval between 3 and 4 kg., it does nct seem probable that the possible sustaining effect of the walls of the sheath can have had any important influence. More probably complications of vicosity, hysteresis, and possibly temperature, or even of adjustment, account for the erratic behavior observed. As such I have refrained from pursuing it further.

32. The same. Brass.—To estimate how far the above work may be in error, owing to inadequate rigidity of apparatus, the hard-rubber rods were replaced by brass rods of about the same size. Their constants were

$$L=2.35 \text{ cm.} \qquad 2r=0.375 \text{ cm.} \qquad A=0.110 \text{ cm.}^2 \qquad R=10.3 \text{ cm.}$$
$$R'=7.0 \text{ cm.} \qquad i=45^\circ$$

An example of cycles obtained in this way (after initial loading) may here be given:

Load	1	2	3	4	5	4	3	2	1	kg.
$10^4\Delta N$	0	6.5	11.0	12.0	12.5	10.5	9.0	7.5	1.5	cm.

One may infer that below 2 kg. there is some dislocation, but very little is shown by the residual loop above 2 kg. Moreover, the loop is not a case of hysteresis, as is clear from the inverted march of values. Hence the mean value $\Delta N/\Delta P=0.00016$ cm. may here be accepted, from which $E=5.5\times 10^{11}$ may be computed. This result is too low; but with a micrometer reading to but 10^{-4} cm., limiting the displacement interferometer, a value even as good as this is hardly expected. It indicates the degree of deficient rigidity

of the apparatus which is not adapted for brass rods as thick as this, as well as the sectional irregularity of stress. If x be the true micrometer displacement and x' that due to apparent yielding of the apparatus and sectional discrepancies, and if C is a constant and 10^{12} the modulus of brass, we may write (since $\Delta N = x + x'$)

$$E = 10^{12} \frac{C/\Delta N}{C/x} \text{ or } x = E\Delta N/10^{12} = 0.00009 \text{ cm. per kg.}$$

while the yield, etc., of parts is thus equivalent to 0.00007 cm. per kg. In any case, therefore, the individual fringes would have to be used in a rigid apparatus for rods of this relative thickness (0.375 cm.). But there is, of course, no difficulty in inserting thin brass rods, taking advantage of the convenience of displacement interferometry. We may also conclude that the insufficiency of the apparatus lies well within the smallest division (10^{-4} cm.) of the micrometer.

Further experiments were now made by loading the rod with 2.5 kg. and making observations in triplets between 2.5 and 4.5 kg. The results came out equally unsatisfactory, being

$10^5 \Delta N/\Delta P =$	18	26	25	26
$10^{-11} E =$	5.1	3.4	3.5	3.4

Between these and the preceding results there may have been some dislocation, such that a smaller part of the sectional area was strained in the latter case, resulting in a decrease of E; but an error in reading of 10^{-4} cm. would also account for it.

To test the preceding values experiments were now made on a hard drawn-brass rod much thinner than the preceding. Its dimensions were

$$L = 2.5 \text{ cm.} \qquad 2\gamma = 0.205 \text{ cm.} \qquad A = .033 \text{ cm.}^2$$

To accommodate this rod a sleeve was turned, telescoping into the larger sheath b, figure 34, and holding the new brass rod within loosely. From triplets of observations between loads of 2.5 and 4.5 kg. the values of E came out as follows:

$10^5 \Delta N/\Delta P =$	149	161	123	120
$10^{-11} E =$	4.2	3.9	5.1	5.2

As this is but half the usual modulus of brass, and as $\Delta N/\Delta P$ is relatively large, the discrepancy must be attributed to yielding or other dislocation within the apparatus.

Tests made for hysteresis showed no effect beyond the possible errors of observation. The mean results obtained from rising and falling loads were, for instance,

$\Delta P =$	1.5	2.5	3.5	4.5 kg.	1.5	2.5	3.5	4.5 kg.
$10^5 \Delta N =$	137	128	121	114	136	127	121	114 cm.
$10^{-12} E =$		0.33	0.44	0.52		0.34	0.47	0.48

The low value of E when the smaller pressures are applied is again the result of irregular or insufficient contact of the rod with the abutment of the apparatus. Much greater pressures are apparently needed to secure an adequately fixed seat of so rigid a body as the brass rod.

33. The same. Glass.—Glass rods of about the same size as the sheath were next tried, the dimensions being

$$L = 2.3 \text{ cm.} \qquad 2r = 0.36 \text{ cm.} \qquad A = 0.102 \text{ cm.}^2$$

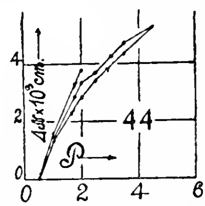
Observations were made in triplets for loads between 2.5 and 4.5 kg. The following results are examples:

$10^5 \Delta N/\Delta P =$	70	75	65	69	69 cm.
$10^{-11} E =$	1.4	1.3	1.5	1.4	1.4

A film of pitch was then placed between the end of the glass rod and the cast-iron abutment. The apparatus was heavily loaded for some time to squeeze out all superfluous cement. The triplets measured between 2.5 and 4.5 kg. of load showed the same order of value, viz,

$10^5 \Delta N/\Delta P =$	77	70	72	69	65	65 cm.
$10^{-11} E =$	1.2	1.4	1.3	1.4	1.5	1.5

The results throughout in very different adjustments are thus remarkably consistent, but they are all enormously too low, probably not more than about one-third to one-quarter of the true value of E. These unsatisfactory results were not expected, because the modulus is only about half that of brass. The sectional distribution of stress is thus even less uniform in case of glass.



Experiments were also made in cycles, but the early results, though markedly looped, were not quite trustworthy. The following values are of later date and better, remembering that the micrometer reads to but 10^{-4} cm.:

$\Delta P =$	0.5	1.0	2.0	2.5	3.5	4.5	3.5	2.5	2.0	1.0	0.5 kg.
$10^5 \Delta N =$	0	130	285	345	440	530	475	373	335	155	20 cm.

They are given in figure 44 and show both in the curved loci and irregularity of detail, that the true elastics of the glass are masked by the incidental discrepancies.

The values of E thus obtained for glass, though consistent, are necessarily too small. It appears, therefore, that a brittle solid like glass does not conduct stress uniformly if the sectional area is relatively large as compared with the length. Thinner rods were next provided, loosely fitting the sleeve already used for brass rods. The new dimensions were:

$$2L = 4.84 \text{ cm.} \qquad 2r = 0.185 \text{ cm.} \qquad A = 0.027 \text{ cm.}^2$$

The rods were placed in the interferometer and the compressions observed, as usual in triplets, between loads of 2 and 4 kg. Most of the individual

contractions were again remarkably consistent. Consecutive mean results for instance are:

$10^5\Delta N/\Delta P=$	152	156	153	135	132	137	168	168	152	147 cm.
$10^{-11}E=$	2.5	2.4	2.5	2.8	2.8	2.7	2.2	2.2	2.5	2.6

As the rods were drawn and not ground true and fitted the sleeve loosely, the occurrence of some dislocation in the data obtained in the three different series is perhaps inevitable. Again the maximum difference of ΔN is within 4×10^{-4} cm. and with a micromter reading to 10^{-4} cm. some of this is observational error. The results, however, show conclusively that even when the glass rod is apparently thin enough as compared with its length, the actual value of the modulus is not obtained. Some outstanding discrepancy has escaped detection.

34. Same. Steel.—A final test of the degree of insufficiency of the apparatus was made by using steel rods loosely fitting the original sheath. Their dimensions were:

$$2L=9.2 \text{ cm.} \qquad 2r=0.37 \text{ cm.} \qquad A=0.107 \text{ cm.}^2$$

Such rods are, of course, far too rigid and thick for the apparatus and the displacement per kilogram would be but

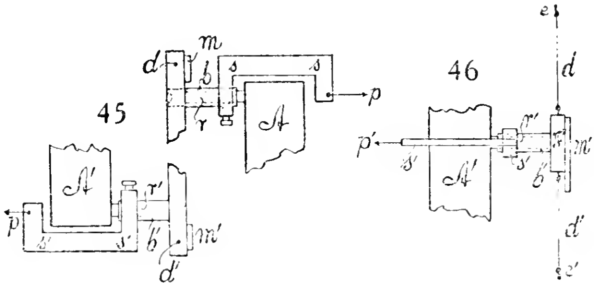
$$\Delta N/\Delta P=10^{-6}\times 44 \text{ cm.}$$

less than the mean wave-length of light, while the micrometer registers but 10^{-4} cm. It would be necessary, therefore, to work with fringes in any case, even if the apparatus were sufficiently rigid, etc., to guarantee such small displacements. In the best results from triplets between 3 and 4 kg., $10^5\Delta N/\Delta P=22$ cm., equivalent to $10^{-12}\ E=0.4$, could not be improved, and yet this is about 5 times too small. Steel rods less than a millimeter thick would have to be used if a trustworthy value of E were aimed at.

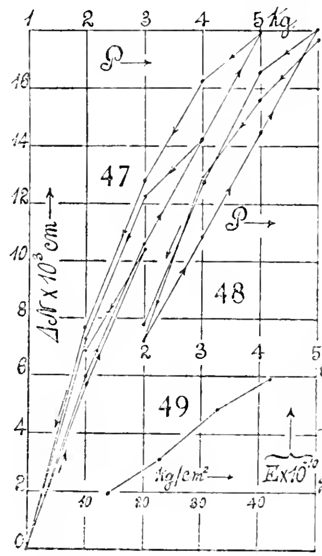
35. Modifications of apparatus.—The above apparatus failed in case of rods of high rigidity and insufficiently reduced sectional areas. Brass and steel rods 2 cm. long will have to be at least as thin as 1 mm. in diameter if the data for E are to be trustworthy. There is, in other words, a source of error in the apparatus itself, by which $\Delta N/\Delta P$ is incremented; and this is to be sought in the method by which stresses are applied. Though each rod is supported at one end by the friction at its contact with the abutment A, figure 28, and at the other by the thread of the bifilar suspension, this support is not guaranteed to the extent required. The vertical bifilar stress yields in its relation to the much larger horizontal stresses of the loads, so that the bar F undergoes independent slight rotations around vertical and horizontal axes of its own. This is evidenced by the fact that the fringes sometimes change their inclination, or the two white slit-images may to a small degree lose their full coincidence (flexure).

The occurrence of the hysteresis loops in case of hard rubber may be

anticipated, *a priori*, although the amount of this quality is also modified by the yield of the apparatus. If hysteresis is marked, the moduli E would naturally increase with increasing loads as the top ends of the loops become relatively more horizontal. One would expect, however, that the curve with continually increasing loads should be straight, and the curvature (concave downward) of the above graphs is probably introduced by the apparatus.



I therefore modified the apparatus as shown in figure 45 (plan) by attaching the offsets ss and $s's'$ firmly to the sheaths b and b' (holding the rods r, r'), by means of set screws. The pull of the pulley strings p and p' is to be coaxial with the respective rods r, r'. FF' is the bar holding the auxiliary coplanar mirrors m, m' and is supported by the bifilar threads attached at d and d'. This bifilar is now also to be advantageously modified (fig. 46, side elevation) by doubling it and attaching the tense fibers d,d', above and below at e and e' to corresponding standards. There are such pairs of fibers at either end of FF'. Thus the bar is held in place in the absence of the stress along p', and this should be applied in such a direction as not to disturb the reflection from the mirrors m', m.



36. Observations.—Cycles obtained with the new apparatus are given in figures 47 and 48. They are in many respects satisfactory, showing the hysteresis loops and the gradual yielding of the viscous solid to stress. In spite of all precautions, however, the slit-images separated very slightly, proving that the rigid attachment of the offset ss (figs. 45, 46) promotes flexure as the result of small cross-torques. The loose offset which can not convey flexural torque is in this respect preferable. Of the values of the modulus computed from the pressure—ascending branch of the cycles—the following data may be recorded by way of example (hard-rubber rod, $2L=5.05$ cm.· $2r=0.37$ cm.; $A=0.107$ cm.^{2}).

P	$10^5\Delta N/\Delta P$	$10^{-10}E$
kg.	*cm.*	
2–4	362	2.7
3–5	360	2.7
1–3	530	1.8
2–4	412	2.4
3–5	367	2.7
1–3	522	1.9
2–4	427	2.3
1–3	520	1.9

The modulus thus increases definitely with the stress, except in the first case; but not so much as in the case of the measurements made in triplets. These were also carefully observed and appeared as follows:

P	$10^5\Delta N/\Delta P$	$10^{-10}E$	Kg./cm.² (load)
4–5–4 kg.	280	3.5	42
5–4–5	173	5.6	
4–5–4	165	5.9	
4–3–4	315	3.1	33
3–4–3	200	4.9	
2–3–2	310	3.1	23
3–2–3	310	3.1	
1–2–1	485	2.0	14
2–1–2	495	2.0	

The rod, therefore, grows apparently more rigid as a result of this alternation of stress, particularly at high loads. The same result is seen in the individual readings of the triplets. If the last results are taken, the curve (fig. 49) results. They make a series much like figure 39 for similarly shaped hard-rubber rods, and the following ratio is of the same order as the above:

$$R=\frac{E}{\text{kg./cm.}^2}=10^{10}\times 0.14$$

With a total change of apparatus the successive increments of rigidity for the same step of loading were again investigated with similar results, as follows:

P	$10^5\Delta N/\Delta P$	$10^{-10}E$	Kg/cm.²	P	$10^5\Delta N/\Delta P$	$10^{-10}E$	Kg./cm.²
4–4–4	275	3.5	42	2–1–2	600	1.6	14
5–4–5	173	5.6		1–2–1	470	2.1	
4–5–4	160	6.1		2–1–2	455	2.1	
5–4–5	165	5.9		1–2–1	465	2.1	

37. Apparent yield within the apparatus.—It remains to determine how far the above apparatus may be made rigid. With such an end in view, steel rods of the dimensions

$$2L=5.03\text{ cm.} \qquad 2r=0.37 \qquad A=0.107\text{ cm.}^2$$

were put in the sheath. In this case (P in kg.), if $C=98$ is the constant of the apparatus,

$$10^{-6}E=C/(\Delta N/\Delta P) \quad \text{or} \quad \Delta N/\Delta P=49\times 10^{-6}$$

about one-half the smallest division of the micrometer, or a little over one fringe (1.2), is all that may be looked for. The results with the *fixed* offset were larger, 3 to 4 fringes; or at the micrometer

$$P=4-5-4 \text{ kg.} \qquad 10^5\Delta N/\Delta P=10 \text{ to } 30 \qquad 10^{-12}E=0.5 \text{ to } 1.0$$

Thus E is about one-quarter of the true value and three-quarters of the displacement is apparently in the apparatus. Under lower loads the conditions are not essentially different. I obtained in the triplets, for instance,

$$P=1-2-1 \text{ kg.} \qquad 10^5\Delta N/\Delta P=33 \text{ to } 50 \qquad 10^{-12}E=0.2 \text{ to } 0.3$$

largely influenced by errors in setting the micrometer, and the increased uncertainty of the seat of the rod.

The fixed offsets (ss, figs. 45, 46) were now removed and replaced by the loose offsets with conical plugs and sockets (figs. 28, 34). Observations were made in triplets as before, showing

$P=4-5-4$ kg.	$10^5\Delta N/\Delta P=33$ to 34	$10^{-12}E=0.23$ to 0.30
$1-2-1$	27 to 40	0.24 to 0.36

These results are identical and substantially the same as the preceding set. In a series of decreasing pressures I found

$P=$	5 kg.		4 kg.		3 kg.		2 kg.		1 kg.
$10^5\Delta N/\Delta P=$	195		155		125		75		0
$10^{-12}E=$		0.24		0.32		0.20		0.13	

so that it is not until all but 2 kg. are removed that the apparatus registered a yield or dislocation of seat. In case of the loose pulley there was no flexure of the bar F; the slit-images remained in coincidence, and the fringes strong and clear at all loads, without changing their inclination. Hence, since the above mean $\Delta N/\Delta P=0.00037$ cm., and as $x=E\Delta N/10^{12}=0.000098$ cm. is to be ascribed to the steel, 0.00027 cm. per kg. may be considered as yielding coefficient of the apparatus; i. e., about 0.0003 cm. should be deducted from all deflections per kilogram of load.

The endeavor was now made to ascertain where this yield is to be located. It was found that by loading the pulley standards b,b' (figs. 28 and 29) no appreciable displacement of fringes was produced, provided the load was the same on both sides b and b'. This is the case in the above experiments. On loading b or b' alone, however, the yield was quite marked, showing $10^5\Delta N/\Delta P=25$ in case of loads of 1 and 2 kg. But this non-symmetrical method of loading is never used.

In a further examination of the apparatus it was found that vertical or longitudinal slight pushes on the bed-plate (tripod) were ineffective, but that a cross-push, even if a mere touch, as this directly tends to change the angle α of the bar F, was very appreciable. Somehow, through the inter-

action of the bifilar and the load, stress or spurious torque about the vertical (hence $\Delta\alpha$) is introduced; but it is extremely difficult to state just how when all loads are symmetrically vertical. To obviate this the permanent parts of the apparatus should be cast in one piece.

An interesting corroboration of these observations is given by the pendulum oscillations of the loads. These produce (even for loads as small as 1 kg.) marked vibration of fringes if the load vibrations are transverse to the rays, while vibrations are ineffective if longitudinal (in direction of the rays). Noticing the behavior of fringes for the first time, I supposed that the centrifugal accelerations introduced by the vibration might be the cause. In fact, if the length of the compound pendulum (load w, figs. 28 and 29) treated as a simple pendulum is L and its mass M, the centrifugal force at any displacement s, corresponding to the displacement velocity v, is

$$F = \frac{Mv^2}{L}$$

But if $s = A \sin \omega t$, then $v = A\omega \cos \omega t$, and thus

$$F = MA^2 g \cos^2 \omega t / L^2$$

since $\omega^2 = g/L$. This force is a maximum at $t = 0$ sec., or $F_0 = Mg \,.\, A^2/L^2$. If $A = 1$ cm. and $L = 15$ cm., F_0 is but $\frac{1}{225}$ of the weight of the load Mg, say a maximum not exceeding 25 grams of weight. Since the whole displacement is but a few fringes, this small fraction could not be discernible with a body like the steel rod above. Thus the alternating transverse stress produced by transverse swinging alone can account for the observed effect.

38. Ocular micrometer. Collimator micrometer.—These methods of measuring the displacement of fringes have been discussed in Chapter I, § 4, 5. It was of interest to test them here. The scale on the ocular plate inserted divided the field width into 100 parts, the division being in 0.1 mm. The distance apart of the achromatic fringes was a little more than this (1.4 cm.) The correspondence could be made exact by rotating the auxiliary plate (bar F, fig. 28) about a vertical axis slightly, but the adjustment was not necessary here. These excessive tenth millimeters thus correspond roughly to an elongation $2\Delta l$ of both rods, equal to the mean wave-length of light, or more accurately,

$$2\Delta l = \frac{R'}{R}\lambda$$

where $2R$ is the distance apart of the interfering rays ab (fig. 27) and $2R'$ the distance apart of the rods rr'.

If the size of fringes is known on the ocular micrometer, they may be counted by their displacement along it, since the central achromatic fringe is always distinguishable and serves as an index. But the width of fringes, if the laboratory is not quiet, is hard to measure, for they quiver or vibrate. It is easier to express the displacement Δe of fringes in the ocular in terms of ΔN, the corresponding displacement of the micrometer of the interferometer.

By direct comparison the following values were found:

$10^3\Delta e=$	460	480	405	460 cm.
$10^5\Delta N=$	145	165	125	155 cm.
$10^4\times$ratio$=$	32	34	31	34

The mean value is $\Delta N/\Delta e=0.00328$, or $\Delta e/\Delta N=305$. The variable Δe is thus over 300 times larger than ΔN.

The cyclic experiments with the steel rods previously used showed a lack of coincidence of the ascending and descending graphs which must here be spurious. I obtained, for instance,

$\Delta P=$	1	2	3	4	5	4	3	2	1 kg.
$10^3\Delta e=$	0	90	170	250	330	275	190	110	0 cm.

As the ascending graph is fairly regular, the apparent modulus may be found from it. Three such series gave mean values of Δe, from which ΔN, Δl, and E were computed as follows:

$10^3\Delta e/\Delta P=$	85	81	86 cm./kg.
$10^5\Delta N/\Delta P=$	28.1	27.0	28.4 cm./kg.
$10^5\Delta l/\Delta P=$	14.6	14.0	14.8 cm./kg.
$10^{-10}E=$	0.35	0.36	0.34

To find Δl, the elongation of each rod, the equation

$$\Delta l=\frac{R'}{R}\frac{\Delta N\cos i}{2}=0.52\ \Delta N$$

suffices, since $R'/R=1.47$ and $i=45°$. The values of $E=97.8/(\Delta N/\Delta P)$ are of the same low order, scarcely one-fifth of the true value already found; i. e., they merely indicate the deficient rigidity of the apparatus.

In the last series of triplets made under different circumstances by aid of Δe, the mean values were

Mean load	15	25	35	45 kg.
Mean $E\times10^{-12}$	0.28	0.35	0.46	0.51

a steady progression, with remarkable consistency throughout; but it would require a load of 20 to 25 kg. to reach the true modulus for steel. So also E increases in successive triplets to a limit; as, for instance, at the loads 4 to 5 kg.,

$$E\times10^{-12}=0.40,\quad 0.49,\quad 0.54,\quad 0.52$$

39. Summary.—It has been found that a spurious displacement of 3 to 6 fringes per kilogram of the load, apparently within the apparatus, has not only not been overcome in the above form of construction, but the cause of this residual discrepancy could not be definitely ascertained. Very slight flexures or torsions of the bar F (fig. 28) by the stresses would account for it; but in such a case the slit-images should lose coincidence, and this has not been typically the case. A slight rotation of the whole apparatus around

a vertical or a horizontal axis normal to the rays from the interferometer would also contribute to the error in question. Yet the method of loading does not seem to admit of such stress, unless the weight supported on the single leveling-screw g of the base (figs. 28, 29) nearer the interferometer, is at a disadvantage as compared with the load supported on two screws remote from the interferometer. Moreover, this apparent yield is consistent, increasing proportionately to the load, so that it is difficult to separate it from the true strains. The only way of counteracting these difficulties is to remodel the apparatus, casting it as a single massive piece of metal, and providing other means (precision slides) of applying stress. The pulley-offset device used must be regarded as improvised, for it is here that fictitious strains undoubtedly enter.

Apart from these considerations and in view of the consistent presence of the discrepancy, its cause might be sought in the unequal sectional distribution of stress from end to end of the rod.

50

Such an effect, moreover (i. e., any unevenness of the seat of the rod r, fig. 50), is not unlikely to produce flexure. Thus the stresses PP' in the figure, acting on the left edge, would tend to bend the rod to the right at the middle. This would also be a purely elastic deformation and behave as such. Again, if in figure 50, P and P' act at diagonally opposite corners, the rod is subject to shear.

There is still another possible explanation of the discrepancy in terms of friction, which may be adduced. Appreciable friction effects at the pulleys are improbable; but at the conical points of the offsets, as they engage the corresponding conical sockets, displacement involving marked friction is not excluded. If, therefore, c is the coefficient of friction and W the weight applied, and if we write $N=CW$, where C is a constant, the effect of additional weight dW may be written

$$dN=C(1-c)dW \text{ or } \Delta N=C(1-c)\Delta W$$

when the load is increased; and similarly,

$$-\Delta N=C(1-c')\Delta W$$

when the load is being gradually decreased. If the effective coefficients are not equal, the observations would therefore correspond to two lines of different slopes and the passage from one to the other would invariably be hysteresis-like, even if there is no such quality appreciable in the elastic solid under observation. Also, the phenomenon would increase in magnitude as the deformations are greater, giving a result very similar to the above observations. True, whether the friction effect occurred in a symmetrical or regular fashion or not, it would not have been eliminated in the triplets of observations made. It is quite conceivable that if a weight W added has produced cW less stress than is implied, the removal of that weight would deduct $c'W$ less stress than implied, or even for larger c' deduct no stress at all. Hence in such triplets ΔW would be much too large or ΔN much too small

and therefore E too large. The effect would be a marked increase of E with the load such as occurs above.

As a result of the residual difficulties enumerated, rods whose modulus is of the order of 10^{12} can not be expected to show trustworthy behavior, unless their diameters are less than a millimeter for a length of about 2 cm. in a suitable protecting sheath. Rods whose moduli lie markedly below 10^{11} may be used when the diameter is 3 or 4 mm. It is for these materials, i. e., the large class of well-developed organic solids, that the apparatus has been devised. From them, moreover, in view of the accentuated deformations, interesting information bearing on the elastics of bodies as a whole may be expected. Viscous phenomena in their entirety, including hysteresis, have a close analogy to the condensation of a vapor. To condense the instabilities to molecular aggregates of small volume takes more pressure than is necessary to release them; i. e., to evaporate them, as it were. If one considers the viscous deformation (*cæt. par.*) as decreasing at a very rapid rate through infinite time, the hysteresis may be considered as an integral part of the viscous phenomenon. The viscous after-effect may then be explained as due to condensation of configurations made unstable or evoked by the heat motion within the body, whereas all the instabilities present at a given time in relation to the applied stress are swept away in the hysteresis phenomenon.

A great many experiments were now made to endeavor to locate the seat of the yielding within the apparatus. Thus the bifilar was variously attached independently of the weighting appurtenances, the base was clamped and screwed down in different ways, a new rigid bar FF was constructed, etc.; but all these attempts failed to eliminate the discrepancy. Pull on the framework and distribution of weights upon it produced no displacement of fringes. Only when weights are placed on the scale pan does yielding (real or apparent) occur; so that it must in some way be connected with the offsets. Replacing the conical ends by sharp darning-needle points was not advantageous.

Somewhat improved results appeared when the bifilar threads were replaced by brass strips about a foot long, a half inch broad, and one-sixteenth inch thick, care being taken to insert them without stress. An example of a cycle (e referring to the ocular micrometer) with steel rods may be given ($2L=5.03$ cm., $2r=0.37$ cm.).

$P=$	5	4	3	2	1	2	3	4	5 kg.
$10^3\Delta e/\Delta P=$	85	75	60	38	0	25	50	70	85 cm.
$10^5\Delta N/\Delta P=$	170	150	120	76	0	50	100	140	170 cm.

The results of many similar experiments all consistently indicated the same order of yield within the apparatus as before. Data would have been much smoother but for the air-currents in a steam-heated room, which caused the fringes to vibrate. The yielding is usually excessive at the lower loads.

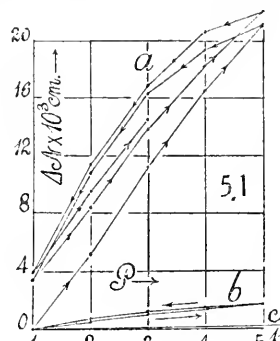
A final test was made by replacing the slit of the collimator by a glass scale (Chapter I, § 5). This is distinctly seen in the telescope and supplies

an even more sensitive micrometer, with the advantage that the telescope may be shifted, there being no fiducial line within it. If $\Delta\epsilon$ refers to displacement of fringes measured along this scale, the standardization showed that $10^3\Delta N/\Delta\epsilon = 101$. Triplets for the loads 1-2-1 kg. gave a mean value of $\Delta\epsilon/\Delta P = 0.25$ cm., whence $10^4\Delta N/\Delta P = 2.51$ cm. and $E = 0.39 \times 10^{-12}$ as above. The change is cyclic, the graphs are curved. The mean E, even above 3 kg. of load, will not exceed 0.3×10^{12}. Triplets gave in succession

$P =$	$1-2-1$	$2-3-2$	$3-4-3$	$4-5-4$	kg.
$10^3\Delta e/\Delta P =$	27	17	13	9	cm.
$E \times 10^{12}/\Delta P =$	0.20	0.29	0.38	0.54	cm.

which is no marked improvement over what has preceded.

There seems to be little hope of further increasing the effective rigidity of the apparatus. Measurements for E will therefore have to be made differentially. For this purpose the constants of the apparatus (here with elastic brass bifilar) are to be determined by a relatively thick steel rod, as has just been done. An example of this has been put in figure 51, a, showing the behavior of a fresh, hard-rubber rod treated in successive cycles of loads between 1 and 5 kg. Hysteresis and viscous deformations appear very clearly, as usual. The mean moduli for the ascending branches will not much exceed 2×10^{10}, and in triplet observations they ran from this to about 6×10^{10} for loads up to about 50 kg. per cm.^{2}, also in the manner found above. As a contrast to these results the behavior of the steel rod of the same dimensions ($2L = 5$ cm., $2r = 0.37$ cm., nearly), for which data have just been given, is also inserted in the lower curve b of the same figure. Both curves are cyclic, but on an enormously different scale, even though the true steel moduli can not be approached to more than one-quarter. The true steel line is shown at c. We admit that the hysteresis in curve b is possibly in the apparatus, which is a compound elastic body; but it is hardly plausible that the hysteresis of curve a can be similarly explained away. The apparatus discrepancy is given in b.



CHAPTER IV.

EXPERIMENTS IN GRAVITATION.

I. GRAVITATIONAL ATTRACTION.

40. Introduction.—The ease with which the rectangular interferometer admits of the measurement of small angles induced me to adapt an apparatus with reference to it, for the measurement of the Newtonian constant. In addition to the usual system suspended in air, I also tested a floating system. Accurate work is scarcely to be expected in this laboratory, where temperature variations and the agitation of the room conflict with the condition of its obtainment. But the trial is nevertheless interesting and the final work may be done elsewhere.

41. Equations.—The old bifilar system has more recently been superseded by the quartz fiber of Boys. We should have in systems of the same length L under like torque T per radian in the two cases, the modulus

$$\mu = T/\theta = mg\; ll'/L = \pi n r^4/2L$$

where mg is the weight of the needle, ll' the spacing (above and below) of the bifilar, n the rigidity, and r the radius of the quartz fiber. If $n = 5 \times 10^{11}$, $g = 981$, this relation reduces to

$$m\,l\,l' = 8 \times 10^8 \times r^4, \text{ nearly}$$

If $r = 10^{-2}$ cm., $m\,l\,l' = 8$, so that for $l = 0.1$ cm. and $l' = 0.5$ cm., $m = 160$ grams would give equivalent elastic efficiency. The bifilar would be superior, as the quartz fiber could not carry this load. If $r = 10^{-3}$ cm., $m = 0.016$ grams only could be used in equivalence, quite apart from the torsion coefficient of the silk fiber of the bifilar. Here the quartz fiber would be superior, as so small a load implies a correspondingly small gravitational force.

In relation to the gravitational experiment, if μ is the modulus in either case, we would have in succession

$$f = \gamma Mm/d^2 \qquad \Delta\theta = \frac{\cos i}{2R}\Delta N \qquad T = fl'' = \mu\Delta\theta$$

where $i = 45°$ is the angle of incidence and $2R$ the distance apart of beams of light, ΔN the micrometer displacement of the interferometer, fl'' the force couple of the torsion balance. Hence

$$\gamma = \mu\,\frac{d^2}{Mm\,l''}\,\frac{\cos i}{2R}\,\Delta N$$

If we take the first case ($l = 0.1$, $l' = 0.5$, $L = 50$) for the bifilar $\mu = 2 \times mg$

$\times 10^{-3}$ (since the load of the bifilar is $2m$ if m is the mass of each ball) and put $M = 10^3$ grams, $m = 1$ gram, $2R = 10$ cm., $l'' = 30$ cm.,

$$\gamma = \frac{d^2}{30 \times 10^3 \times 1} 2 \frac{0.71}{10} \Delta N = 4.8 \times 10^{-6} \times d^2 \times \Delta N, \text{ nearly}$$

or

$$\Delta N = 6.7 \times 10^{-8} / 4.8 \times 10^{-6} \times d^2 = 0.015 d^2$$

If d is estimated as 3 cm., then $\Delta N = 0.0016$ cm. With the given interferometer and reasonable estimates as to the other magnitudes, one should therefore obtain nearly 40 achromatic fringes (even with the bifilar as stated) for the attractions of 1 kg.

There would be no gain, in case of the bifilar, by increasing the mass m at the ends of the needle; for the modulus of the bifilar increases as m, which would therefore cancel the m in the denominator of the expression for γ. But if the system is floated in water, m may be increased with advantage indefinitely, while the load of the bifilar is kept constant by providing a corresponding float. If this bifilar load is m', we therefore have

$$\gamma = \frac{d^2}{M m l''} m' g \frac{l l' \cos i}{L \; 2R} \Delta N$$

Inserting the data of the apparatus of the next section,

$d = 5$ cm. $\quad l'' = 30$ cm. $\quad M = 10$ g. $\quad m = 30$ g. $\quad m' = 2$ g.
$ll' = 0.05$ cm.^{2}. $\quad L = 50$ cm. $\quad i = 45°$ $\quad 2R = 10$ cm.

$$\gamma = \frac{25}{30 \times 10^3 \times 30} 2 \times 10^3 \frac{0.05}{50} \frac{0.71}{10} \Delta N = 3.9 \times 10^{-6} \Delta N$$

or

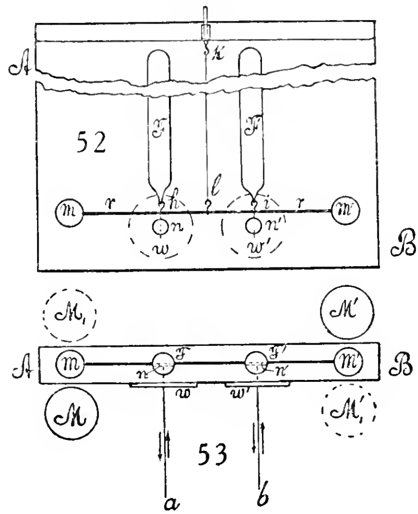
$$\Delta N = 0.017 \text{ cm.}$$

per attracting kilogram. With a reasonable size of float there is no difficulty in increasing the attracted mass m to over 60 grams and the attracting mass (with a slight increase in d) to 10 kg., so that values of ΔN of the order of a millimeter are not out of the question.

The difficulty with the method lies in the simultaneous increase of the period of vibration of the needle, and this seems fatal; but I thought it worth while, nevertheless, to give the method a trial.

42. Observations. Floating system.—Figures 52 and 53 represent the floating needle submerged in the narrow trough AB provided with two windows of plate glass $w\ w'$, through which the interfering beams enter and leave, nearly at right angles to the coplanar mirrors n and n' attached to the needle. This consists of a tube of aluminum, about 30 cm. long and 6 mm. in diameter, with balls at the ends m and m', pairs of 30 grams to 60 grams each being admissible. The two hooks h and i carry the floats F, F', test tubes as much

as 15 cm. long and 2.5 cm. in diameter, but to be changed in capacity with the balls m, m' used. The stems of the hooks, screwed and sealed into the tube rr, carry the mirrors n, n'. The needle is suspended from a torsion-head by the bifilar of silk fiber kl, also completely submerged in the water-bath. The hooks k and l are provided with a thin flat sheet-metal link, by which the bifilar may be appropriately spaced.



The needle system is so adjusted as just to float. It is then weighted by 1 or 2 grams to sink it. The weight in water may be measured at K. In view of the weight of needle, the mirrors n, n' may be rigidly connected by a very narrow strip of this plate glass, which facilitates adjustment.

The attracting weights M and M' up to 5 kg. were used in pairs M, M_1 and M', M'_1, on each side suspended by a steel belt from a pulley overhead in such a way that when M, M' are in place M'_1, M_1 are raised out of effective reach.

The chief difficulty was encountered in floating the needle. When this was done the whole tank was fastened in place on the interferometer, as the torsion-head at k is attached adjustably to the tank.

The fringes were found without much difficulty; but they were in incessant motion, owing no doubt to eddy currents produced by temperature differences. After many trials I concluded that measurements would be untrustworthy and further trials were therefore abandoned. The experiment is in fact too difficult for a single observer and would be feasible only in an environment of perfect quiet and constant temperature.

43. Expeditious fringe detection.—In work like the present it is necessary to find the fringes quickly under considerable disadjustment of parts. In this case, if the auxiliary mirror m (fig. 4, Chapter I) can be manipulated, the two images in the telescope T are first made coincident by adjusting M', in which case the rays are parallel as they enter T. The mirror m is then rotated around the vertical and the horizontal axis, until, to the eye, the spots of light coincide locally on the face of M'. Fringes when found by moving the micrometer here are then strong. If mm can not be interfered with, the rays are first made parallel as before by the coincidence of images in T. Thereafter the mirrors M and M' are rotated around a horizontal and a vertical axis *in parallel* (i. e., successively or alternately retaining their parallelism) until the spots of light on M' again coincide, locally, to the eye, or better, when caught objectively on a screen. It is clear from figure 2, Chapter I, that the remoter ray from M is displaced on the screen more

rapidly than the nearer ray from M' and therefore parallelism with local coincidence of rays on M' is generally possible. If the parallel rays T_1 and T_2 which coincide in T are too far apart, no fringes can be found, even when the path-difference is annulled.

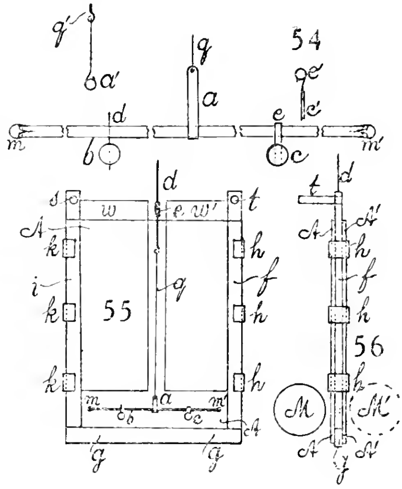
44. Heavy needle in air.—The needle was now deprived of its floats and other appurtenances, provided with somewhat lighter balls ($m=18$ grams each), and suspended in the same float-vessel in air. Since in this case ΔN is independent of m, and as the dimensions were about of the same order given in the example above ($ll'=0.05$ cm., $L=50$ cm., $M=10^3$ g., $2R=10$ cm., $l''=30$ cm.) with a somewhat larger $d=6$ cm., $\Delta N=0.0004$ cm. per kilogram of attracting load was to be expected; i. e., about 10 achromatic fringes.

The adjustment is quite difficult, since the small mirrors must not only be approximately parallel, but must be so spaced and inclined as to receive the component beams and to reflect them through the mirrors of the interferometer. This was accomplished with some patience and the achromatic fringes were found. But here again they proved to be in incessant and relatively rapid motion, so that they swept continually through the field. Though observed for some time, on different days, they were never found sufficiently quiet to admit of the above measurements. It was impossible, in other words, to eliminate the air-currents within the shallow envelope sufficiently to warrant counting at the rate of 10 fringes per attracting kilogram. In the summer I think this would have been feasible, as there was no other drawback militating against the completion of the experiment.

45. Light needle in air.—In view of the failure of the heavy needle, I went to the opposite case of a relatively light needle, weighing when loaded but 1.49 grams. This was made of a rigid shaft of straw (mm', fig. 54) 25 cm. long, the ends being slit slightly into four symmetrical segments each, which receive the two shots m and m', additionally secured with a little wax. The two light mirrors b and c were differently mounted; b on a fine pin d, snugly fitting corresponding perforations in the straw, was thus capable of rotation around the vertical axis (d) and moving up and down slightly. The mirror c, however, was mounted on a thin elastic strip of aluminum, clasping the shaft as shown in section at $e'c'$, and thus capable not only of rotating on a horizontal axis, but of being placed at different distances (moving right and left) from d to accommodate the rays of the interferometer, to which b and c are to be normal. The needle is swung from a quartz fiber q and a strip or hanger of elastic aluminum a (see a', q'), which clasps the shaft. Hence the latter may also be moved endwise to be balanced and rotated around a horizontal axis till b and c meet the rays normally. These operations are completed by trial before the needle is definitely hung, preferably in a broad beam of sunlight. No great accuracy is required.

The shallow case is made of two plates A, A' (figs. 55, 56) of $^1/_8$-inch plate

glass about 32 cm. broad and 45 cm. high, spaced by two strips $i\,f$, 3 cm. wide, of thick (9 mm.) plate glass, reaching not quite from top to bottom. Six steel clips h,h,h,k,k,k, (such as are used for binding pamphlets) held these strips in place and also clasped securely on the outside two wooden strips of about the same width and one-half inch thick. At the top of the wooden strips, two nipples, s and t, of ¼-inch gas-pipe, projecting normally to the strips, served for the hanging of the case and needle from a firm wall-bracket. It would have been preferable to use the wood strips (without the glass strips) clasped between the glass plates A, A' as spacers and hangers at once, and this was eventually done. The bottom of the case is subsequently to be closed from below by a strip of felting gg. To diminish the space within the case two thin cloth-covered wooden boards w,w' are inserted from the top. The quartz fiber q, to carry the needle mm, hangs from a long $^1/_8$-inch brass rod d, which may be raised or lowered in view of the sleeve e, held by a separate adjustable arm without. The rod d must fit the perforation in the cork nicely, so that the former may be smoothly raised or lowered and held in any position by virtue of friction. To swing the needle this is first placed on cork Y's below the opening of the case A,A', the felting $g\,g$ having been removed. The quartz fiber is then lowered on the long stem d, until the lower hook on the fiber is in position to grasp the clasp on the needle mm, still below the case. The needle is then cautiously lifted by raising d until it has the required position relative to the interferometer, about as shown in the figure. After this the felt strip gg is inserted to close the case, and the necessary adjustment made at e and d to swing the needle freely in the restricted space provided for it.



Observations were made with the needle at some length. The quartz fiber used was $L=17$ cm. long; the distance d apart of attracting weights M and attracted weights m was 6 cm. Hence for $M=10$ gr.

$$\gamma=\frac{36}{23\times10^3\times75}\,\frac{5\times10^{11}\pi}{2\times17}\,r^4\frac{0.71}{10}\Delta N=6\times10^6 r^4\Delta N$$

Thus, if $r=10^{-2}$ cm. $\Delta N=1.1\times10^{-6}$ cm. per kg. of M $T=4.6$ sec.,
$r=10^{-3}$ cm. $\Delta N=1.1\times10^{-2}$ cm. per kg. of M $T=465$ sec.

The first case would then correspond to but one-fortieth of a fringe; in the second case there should be 250 fringes per attracting kilogram.

If the tenacity of quartz be taken as 1.5×10^3 kg. per sq. cm. the latter

filament ($r = 10^{-3}$ cm.) should still hold 4.5 grams, or much more than the weight of the above needle (within 2 grams.)

As the moment of inertia of the needle is $2 \times 0.75 \times 13^2 = 253$, and if the modulus of torsion be computed from the slide modulus $n = 5 \times 10^{11}$, the periods would be as given above, the second being nearly 8 minutes. The period of the above needle was estimated at about 2 minutes. This would give a modulus of torsion about 0.7 and make $\Delta N = 67 \times 10^{-5}$ cm. per attracting kilogram; i. e., about 17 achromatic fringes per kilogram were to be expected.

Having mounted the needle as stated, the fringes were found without much difficulty and the image of the wide slit (i. e., the reflected beam seen in the telescope) was almost quite stationary, the light needle being thus adequately damped. But within this virtually stationary slit-image, the fringes (preferably made horizontal) continually wandered up and down, showing that micrometric vibration had not been eliminated. The experiment is a very impressive one, but as the drift is still much larger than the 17 fringes per kilogram to be anticipated, any attempt at measurement is again idle. Whether this drift is due to temperature or to the tremors of the laboratory would be difficult to state. Cutting down the intensity of the interfering rays (which need not be strong) made no difference.

When the case is open above, there is no difficulty in finding the position of equilibrum of the needle symmetrically to the glass walls of the case. This position is assured by the parallelism of the images of the needle in both faces with the needle itself, and their distance from it. When, however, the wooden boards are inserted, the needle does not seem to be in stable equilibrium in the symmetrical position. It tends to move either into one or the other extreme of oblique positions at which the balls touch the plates of the case. Believing the phenomenon to be electrical, I placed a radium tube in the vicinity of the case for some time, but this made no difference. Damp cloth did not change the result. This was particularly true in the earlier experiment, where ¼-inch plates were used for the case in the absence of thin plates, and in which one plate was thicker than the other. One would thus be inclined to interpret the instability as possibly due to the gravitational attraction of the residual disk. Considering the case as that of a point mass confronting an infinite disk, the potential would be $\gamma(C \pm x)$ $2\pi\sigma$ and the force $2\pi\gamma\sigma$ per gram attracted, which is a little above the mass of the ball. Thus, if $\sigma = \rho t$ (σ being the density and $t = 0.1$ cm. the residual thickness of plate corresponding to the surface density σ), the force should be

$$f = 6.3 \times 6.7 \times 10^{-8} \times 3 \times 0.1 = 1.3 \times 10^{-7} \text{ dyne.}$$

Since the lever arm is 13 cm., this makes the torque 1.7×10^{-6} dyne cm.; and if the modulus of torsion of the quartz fiber is 0.7, as estimated above, the deflection should be

$$\theta = 1.7 \times 10^{-6}/0.7 = 2.4 \times 10^{-6} \text{ radian}$$

i. e., about half a second of arc and therefore quite ineffective so far as instability is concerned. Neither does it seem plausible that the needle

striking the glass under the influence of the γ rays of radium or in a damp atmosphere can be charged. Yet the phenomenon which shows itself as an accelerated drift toward either plate is very decided, so that the free position of equilibrium to be obtained by gradually adjusting the torsion-head can not be found.

In view of the importance of this question, I installed a thin quartz fiber of about the same length $L=17$ cm. and found its moment of inertia by attaching to the fiber a brass cylinder $l=3$ cm. long and $m=2$ g. in weight. The period of vibration was found to be 50 to 52 seconds. Hence, as the moment of inertia is $1.5\ \mathrm{g}\times\mathrm{cm.}^2$ the modulus of torsion is $n=0.024$. But even this (by the above method of computation) would not be appreciably attracted

$$(\theta=1.7\times10^{-6}/0.024=7.1\times10^{-5}\text{ rad.})$$

by the glass plates; and yet this tendency is marked.

One may suppose, therefore, that each end of the needle is attracted by the nearer glass plate independently. This will give a superior limit enormously larger than the preceding estimate; for the force is now

$$f=2\times2\pi\gamma\sigma=4\times3.14\times6.7\times10^{-8}\times1.8=1.9\times10^{-6}\text{ nearly, if }\sigma=3\times0.6=1.8$$

the mean thickness of plate bearing 0.6 cm. Hence the torque is

$$10^{-6}\times1.9\times26=4.8\times10^{-5}$$

since the length of needle is 26 cm., and finally

$$\theta=4.8\times10^{-5}/0.024=2\times10^{-3}\text{ radian}$$

Even this excessive estimate, however, fails to account for the result; for the deflection is but a little over a half degree. Moreover, the short brass cylinder in question (length 3 cm.) showed a similar tendency to take oblique positions not corresponding to the torsion-head, as the long needle.

46. Summer experiments.—It is obvious that a fair trial of the apparatus can not be made in an artificially heated room. For this reason experiments in a semi-subterrranean room of the laboratory were reserved for midsummer.

In the summer installation in a subcellar at constant temperature, with a few improvements of apparatus (the mirrors being readjusted, etc.), the needle was without difficulty made to take a stable position midway between the glass plates, subject to the torsion of the fiber. Some trouble was experienced in finding the fringes, owing to incidental causes. The adjustment is made difficult in view of the definite distance apart of the small mirrors to which the breadth of the ray parallelogram must conform. With the micrometer at 45° the latter is very limited in its displacement. I later attached a special micrometer with three identical pairs of parallel V-mirrors (the latter at 90°) similar to the design shown in figure 21. The middle V-mirror is movable in a micrometer. This, when the mirrors are parallel, has the additional advantage of being independent of slight changes of inclination

in the micrometer. The displacement of mirrors is now virtually parallel to the rays and no difficulty in finding the fringes need occur. Naturally the mirrors must be good, there being now four additional reflections in each ray; and this V-micrometer must be accurately adjusted for parallelism of mirrors.

With the fringes found, there is now no difficulty in showing the attraction of gravitation. In fact, an iron brick moved on a small truck, near the shot at one end of the needle, grips these balls very much like a magnet acting on the pole of a magnetic needle. By approaching and withdrawing the iron mass on one side, the fringes could be put in regular and uniform vibration over enormous (relatively speaking) arcs as measured by fringes. Thus in an incidental experiment the micrometer reading was 0.255 cm. with an iron mass near and slow vibration and 0.026 cm. (eventually) with iron mass remote.

Throughout the whole of the experiment the fringes were under the perfect control of the micrometer.

A more systematic experiment was then made by testing the attraction of a lead ball 5.43 cm. in diameter and weighing about $M=950$ grams for the shot (at the end of the needle) weighing $m=0.61$ gram. M was moved on a circular track with stops to a distance of $R=4.24$ cm. (between centers of balls) from the ball of the needle, alternately. The position of the large ball M was reversed every 10 minutes, but the period of the air-damped needle can not have been less than 18 minutes. The case is, then, that of a forced vibration under constant force and a large logarithmic decrement. The observations are given in table 4, the reading being made every minute, beginning with the equilibrium position (M in the neutral position). If these data of the displacement x of the mass m are constructed graphically it will be seen that the motion of the needle is nearly dead-beat. The successive arcs of vibration increase, and from the limiting distance between elongations the attracting force could be computed, if the torsion coefficient of the quartz fiber and the logarithmic decrement were known. The limiting arc was not reached, owing to incidental reasons. From static experiments made during hour intervals this elongation was found to be about 0.116 cm., or a departure of the shot m at the end of the needle from its position of equilibrium of 0.058 cm. in response to the attraction of M. If l is the semi-length of the needle (between centers of shots), the micrometer displacement ΔN and the displacement Δx of the mass m are given by the equation

$$\Delta x = l\Delta N \cos i/b = 0.89\,\Delta N$$

where b is the breadth of the ray parallelogram and $i=45°$ the angle of incidence of the interferometer. Thus the micrometer displacement is of the same order as the displacement of m, and if the latter is 0.116 cm., we should have

$$\Delta N = 0.13 \text{ cm.}$$

TABLE 4.—Gravitational attraction. $M=950$ grams, $m=0.61$ gram. R=4.2 cm. Period (damped) about 18 min. Length of needle, 25.2 cm.; weight (total), 1.9 grams.

Adjustment	Time.	$x\times10^3$	Adjustment	Time.	$x\times10^3$	Adjustment	Time.	$x\times10^3$
	min.	*cm.*		*min.*	*cm.*		*min.*	*cm.*
Ball *M* right (from equilibrium)	0	26	Ball *M* right	21	50	Ball *M* right	41	94
	1	31		22	60		42	101
	2	35		23	69		43	111
	3	40		24	79		44	126
	4	46		25	90		*45	139
	5	54		26	98			
	6	58		27	105			
	7	64		28	110			
	8	68		29	117			
	9	72	Ball *M* turned	30	124			
Ball *M* turned	10	77	Ball *M* left	31	124			
Ball *M* left	11	78		32	123			
	12	77		33	120			
	13	74		34	117			
	14	69		35	113			
	15	64		36	108			
	16	61		37	105			
	17	56		38	100			
	18	53		39	96			
	19	49	Ball *M* turned	40	93			
	20	48						

* Out of field of telescope.

As the micrometer reads to 10^{-4} cm., 1/1300 part of the attraction between M and $m=0.61$ gram could therefore be detected; i. e., the attraction of $950/1300=0.73$ gram, or per interference fringe well within one-third of this, for the given quartz fiber, which was not specially selected, and distance R. This is equivalent to the attraction of two tenth-gram masses per centimeter of distance per fringe.

Apart from the measurement of the torsion coefficient of the fiber, there is, however, a real difficulty involved, and that is the occurrence of marked drift in the needle. It is only incidentally that the fringes are found at rest. The chief contributory cause of this is no doubt the occurrence of motion of air around the needle provoked by small differences of temperature, resulting (for instance) from illumination. If the possible accuracy of deflection measurement is to be of any value, therefore, the apparatus must be kept in the dark, except during observation. Fortunately, the achromatic fringes require little light. Even then a closed case which can be exhausted of air is essential, for such radiometer forces as may enter would in any event be differential, seeing that the mirrors are symmetric and the illumination is not subject to alternations like those in the table. It is probable that in case of the above regular method a thicker quartz fiber and a greater distance R would conduce to the best results, since ΔN is the least difficult quantity to determine.

Measurements could not be attempted in time for the present report, but the question may be asked whether it is not possible in the present case to

determine the attractions in terms of the mere acceleration of balls resulting. With an ocular micrometer this would not be difficult, as the fringes move slowly enough so that the position can be sharply specified; but with a needle of long period in vacuum, the screw micrometer would also be available. If there is no damping we may write

$$\gamma Mm/R^2 - ax = 2m\alpha$$

where α is the acceleration, x the displacement of m, and a the torsion coefficient, referred to the displacement of m, t the time, supposing the needle starts from rest, and the gravitational force is applied at $t=0$. If at the outset we may put $x = vt/2 = \alpha t^2/2$

$$\gamma Mm/R^2 = x(at^2 + 4m)/t^2$$

an equation whose interesting feature is that if t is kept very small (which should be possible with an ocular micrometer and the achromatic fringes, a fine quartz fiber presupposed), the term involving t may be neglected and the experiment interpreted as a case of uniformly varied motion, in which

$$\gamma = 2R^2\alpha/M$$

For instance, if $R=5$ cm., $M=10^3$ grams, and $\gamma=6.7\times10^{-8}$, and if $t=100$ sec. is admissible, $\alpha=1.3\times10^{-7}$ cm./sec.^{2} and the distance traversed in 100 sec. would be 0.0065 cm., well measurable on the interferometer, quite so if the work is done reciprocally and the interference fringes are used individually. The theoretical error will be a minimum if m is as large as the fiber can safely carry and t as small as possible. On the other hand, x is independent of m, and if t is to be kept small, the result may be compensated in a large M/R^2. The measurement is thus to consist in keeping the fringes at zero by moving the micrometer screw for the small interval t during which the weight M acts. The constant would then follow from the micrometer reading M and R only, all other quantities entering secondarily as corrections. The experiment seems well worth while.

II. USE OF THE RECTANGULAR INTERFEROMETER IN CONNECTION WITH THE HORIZONTAL PENDULUM.

47. Introductory.—In 1915 and in the reports of the Carnegie Institution of Washington, No. 229, Chapter I, part 2, pp. 30 *et seq.*, I adduced a method for the application of the displacement interferometer to the horizontal pendulum with a graphic exhibit of the results obtained during a series of months. The concave-mirror design by which the spectrum interference ellipses were made available showed a very satisfactory performance, in spite of the fact that deformations of the pier to which the pendulum was attached were local disturbances and excessive in amount. The attainable accuracy was such that for moderate constants in the installation of the pendulum, an inclination of 3×10^{-4} second of arc should have been registered per vanishing interference fringe (ellipse), or about 10^{-3} second per 10^{-4} cm. of displacement of the micrometer. The inclination of the line of suspending pivots was here about $1°$ to the vertical. A smaller angle would have correspondingly increased the sensitiveness.

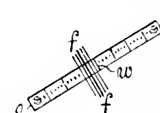
The apparatus, however, required a space about 2 meters long between the extreme mirrors for its installation. This is in a measure a disadvantage, since small changes of temperature in the brackets and supports, as well as in the pier, would interfere with the full realization of the precision of the method. In this respect the rectangular interferometer with an auxiliary mirror is to be preferred; for here all the necessary parts may easily be placed within a distance of 1 foot from the wall of the pier carrying the horizontal pendulum. If the achromatic fringes are used, these are straight and intense, so that photographic methods are available, while for visual observation a gas flame would give sufficient light. The sensitiveness under similar conditions would be slightly smaller, but not enough to cancel the advantages specified.

48. Apparatus.—The old horizontal pendulum formerly described was again used. It was made of thin steel tubing, and in this respect, since its plane was nearly in the meridian, may be subject to change of the earth's magnetic field; but as my object here is merely the trial of a method, these annoyances are of slight consequence.

Figure 57 gives a sectional plan of the pendulum installation and figure 58 a front view of the pendulum *HH* alone, on a somewhat reduced scale. Its general shape is that of an isosceles triangle and the distance from the line of pivots tt' to apex B about 110 cm., while the distance between pivots was 97 cm. The pivot supports SS' are fine screws ending in hard-steel points, which enter a glass-hard steel socket (below) and a steel groove

(above). The line tt' of the horizontal pendulum can thus be given any inclination to the vertical, while the rods p,p' which receive the screws S,S' may be moved mormally to the wall of the pier PP', inward or outward, and clamped to secure parallelism between the pier PP' and pendulum HH'. The apex B of the pendulum is also provided with a clamp, holding vane D submerged in an oil-vat v for damping.

The whole pendulum is inclosed by a flat case CC' of tin plate provided with a plate-glass window at g, through which the auxiliary mirror m of the interferometer may be seen. This is attached to one or two vertical tubes h,h' of the pendulum, adjustably, so that it can be moved up or down, and rotated slightly above a vertical and a horizontal axis.



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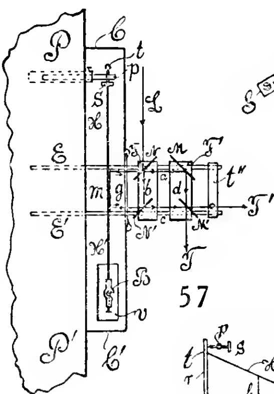

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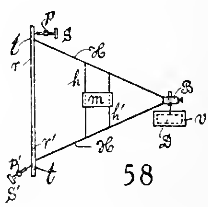

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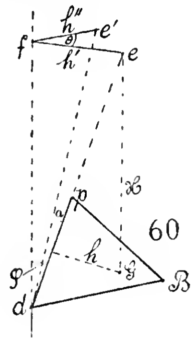
The interferometer consists essentially of the 4 plate-glass mirrors M,M', N, N', all but M being half-silver, the collimator (beyond L) and the telescope at T or T', t'' being a telescope support. The collimated white beam L is thus separated into the component rays $LNmadt$ and $LbN'mcT$, to be observed at either T or T'. M' is on a micrometer slide with the screw normal to the face of the mirror. All mirrors must be capable of slight rotation about horizontal and vertical axes and the silvered faces all lie towards m for compensation of glass paths. The rays leaving M' for T must not only be accurately parallel, but locally (visible as spots of light) nearly coincident, as specified above (Chapter IV, § 43). Otherwise the fringes will be weak or invisible.

The telescope T should be provided with an ocular micrometer (centimeter divided in tenth millimeters) standardized by aid of the sliding micrometer at M', since the main purpose here is the measurement of small angles. Moreover, the image of the wide slit of the collimator adapted to the use of the achromatic fringes should be placed at right angles to them, with the ocular micrometer so placed as to read from end to end of the slit-image. A very fine wire beam across the slit gives the fiducial line relative to the ocular micrometer. Figure 59 shows the general arrangement, SS' being the wide, oblique slit-image, ff' the achromatic fringes, w the image of the fiducial wire across the slit, and ss' the ocular scale. Of course the fringes may be made horizontal or vertical; but this requires much adjustment or else compensation, and is therefore an unnecessary complication of the preliminary work. With this fiducial mark at the collimator (which is permanently out of reach), if the telescope is accidentally shifted, or temporarily removed, it may be

replaced without difficulty. It is the telescope, however, which contains the ultimately fiducial scale, and like the collimator it should be held on a standard t'' suitably attached to the pier. Similarly the mirrors M and M', N and N', fixed in pairs to slides or carriages F,F', are clamped to two parallel horizontal tubes E,E' (⅜-inch gas-pipe smoothed, for instance) anchored in the pier. The highest attainable rigidity in the placement of the mirrors M, M', N, N' and of the telescope is essential. At the outset of the work the viscous yielding of standards and braces is quite apparent. When, as in the present paper, the observations are made at T' and not at T, the telescope is conveniently attached to the slide rods E,E' joined in front at t''.

49. Equations.—In figure 60 let pBd denote the horizontal pendulum in the plane of the diagram and dpe the line of pivots prolonged, terminating in e vertically above the center of gravity G. Let the inclination of de to the vertical be φ, a constant of the apparatus, and suppose a perpendicular h' is let fall from e to the vertical df through d. If, in consequence of a change in the inclination of the pier, the line of pivots passes to de', over a nearly vertical angle α, h' will pass into h'' over a horizontal angle θ. Thus the measurement consists in finding α in terms of the interferometer angle θ. Since these angles are all very small, we may write (Δ being a differential symbol), as shown in the preceding paper,

(1) $$\Delta\alpha = \varphi\Delta\theta$$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But in the rectangular interferometer with an auxiliary mirror, if the distance apart of the rays a and c (fig. 57) be $2R$,

(2) $$4R\Delta\theta = 2\Delta N \cos i = n\lambda$$

where $i = 45°$, ΔN the displacement of micrometer to bring the achromatic fringes back to the fiducial line, and n the number of fringes which pass that line. Hence

(3) $$\Delta\alpha = \varphi\frac{\Delta N \cos i}{2R} = \varphi\frac{n\lambda}{4R}$$

The smallest angle, $\Delta\alpha$, which can thus be measured depends essentially on $2R$, the breadth of the ray parallelogram. There would be no difficulty in making this as long as the line from tt' to B of the horizontal pendulum, i. e., over a meter; but this would necessitate two mirrors m, one at each end. For the present purposes I preferred to use apparatus which I had at hand, in which $2R$ was but 10 cm. and a single mirror could be used at m. Nevertheless, if $n = 1$, the limit of angles measurable, if $\varphi = 0.0175$ radian, or 1° is

$$\Delta\alpha = 0.0175\,\frac{6\times10^{-5}}{4\times5} = 5\times10^{-8}$$

radian per fringe; i. e., about 0.01 second of arc. With an ocular micrometer and well-produced achromatic fringes there is no difficulty in estimating

one-tenth fringe, so that the limiting angle here is to be a few thousandths of a second, even if $\varphi = 1°$, which may also be reduced. By making $2R = 100$ cm. one should therefore be able to reach 0.0001 second per tenth-fringe breadth if φ is $1°$.

Similarly, if ΔN reads to 10^{-4} cm., $\varphi = 1°$,

$$\Delta\alpha = 0.0175 \frac{10^{-4} \times 0.71}{10} = 1.2 \times 10^{-7} \text{ radian}$$

or 0.025 second of arc, with the opportunity of passing to 0.0025 if R is a meter.

Finally, on using the ocular micrometer for moderately sized fringes of say one scale part (0.1 mm. fringes in the ocular), the case is equally promising. A comparison of the two displacements ΔN at M' and Δe of the fringes in the ocular showed

$\Delta e =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9 cm.
$10\Delta^5 N =$	1	35	75	120	155	190	225	265	305 cm.

Fluctuations are due to the motion of the pendulum. Thus the mean value is

$$\frac{\Delta N}{\Delta e} = 0.0038 \text{ or } \frac{\Delta e}{\Delta N} = 265$$

Hence

$$\Delta\alpha = \varphi \Delta e \cos i \,/(265 \times 2R)$$

If $\Delta e = 10^{-2}$ (one scale part) and $2R = 10$ cm., etc., as above,

$$\Delta\alpha = 0.01 \frac{10^{-2} \times 0.71}{265 \times 10} = 2.7 \times 10^{-8} \text{ radian}$$

or 0.005″ per scale part of the ocular micrometer. A few tenths of this may be estimated on the scale. The sensitiveness would be 10 times greater if $2R$ were a meter.

50. Observations.—The interferometer was installed with rather smaller fringes than instanced above and therefore with less sensitiveness, as the inclination of the pier in a heated laboratory would probably run into seconds of arc in the lapse of time. For this reason R was also satisfactory at its small value of $2R = 10$ cm. The angle φ was directly measured, as the inclination of the line joining the points of the pivots to the plumb-line. Under these circumstances the constants given at the head of the table suffice. Since $\Delta\alpha = 0.9\ \Delta e$ seconds, roughly, the tenth millimeters of the ocular scale are about 0.01 second of arc in relation to $\Delta\alpha$ and the fringes were of about the same size. There would have been no difficulty in making them much larger and therefore more sensitive, as they were clear and strong. As it was, there should have been no difficulty of estimating within 10^{-2} second.

The end of the compound pendulum was damped in lubricating oil. This is probably too viscous for refined work, but the purpose here is merely to try out the method.

The earlier observations were discarded, but even after January 14, after which time the apparatus worked comparatively smoothly, instances of displacement within the apparatus required readjustment. These betray themselves in a lack of coincidence of the two wide slit-images (fig. 59) or of the cross-wire w (the slit is really superfluous except as the collimator lens may be so placed as to widen the illuminated field). This is probably referable to the two supports $E, E,'$ which change their parallelism with marked changes of temperature in the room; or it may have been within the apparatus at $M, N, N,' M'$. It is difficult to allow for it, and a reconstruction of apparatus is the only resort.

The illuminant was an electric arc at a distance of about a meter from the interferometer. This was chosen for convenience solely, as the achromatic fringes can be adequately seen with a Welsbach lamp closer at hand.

The observations will for convenience be given graphically. I merely recall that the breadth of the interferometer rectangle was $R=10$ cm.; the inclination of the pivots of the horizontal pendulum about $\varphi=0.01$ radian; the angle of incidence of rays $i=45°$. Hence the change $\Delta\alpha$ of inclination α of the pier will be (if ΔN is the displacement of the mirror micrometer and Δe of the ocular micrometer)

$$\Delta\alpha=\varphi\Delta N\cos i/2R=5.9\times10^{-3}\varphi\Delta e\cos i/2R$$

or

$$\Delta\alpha=4.2\times10^{-6}\Delta e\text{ rad}=0.86\times\Delta e\text{ seconds of arc}$$

Observations were made at about 10 a. m. and at 6 p. m. Variations of α might easily have been recorded during the day, but these are not of interest in their bearing on the present paper. It seemed premature, moreover, to attempt the installation of photographic apparatus, as this would interfere with the visual observation, which is the chief purpose here.

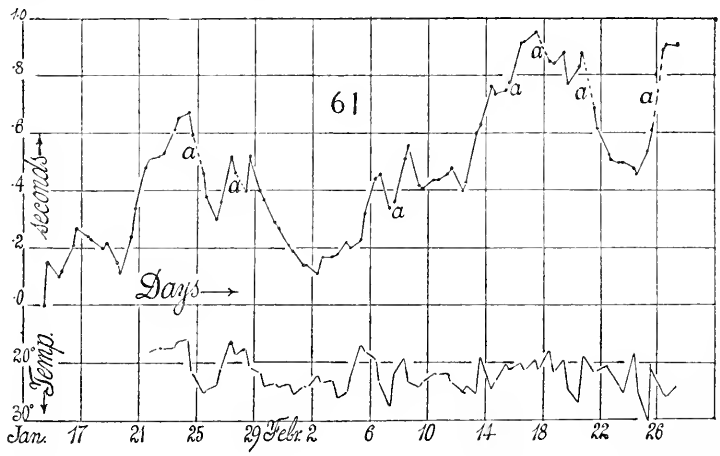
In the graphic figure 61, the observations at which adjustment of slit-images was necessary are marked a.

a. This effect was always in the same direction; i. e., indicating that if no readjustment had been needed the curve between January 13 and February 27 would have risen as a whole more rapidly than the data actually inscribed show. As a mere matter of convenience the curve has been made continuous irrespective of readjustments (on the average a rise of about 0.1 second each). In this way the whole change of inclination within the 45 days of observation did not exceed a second of arc and should therefore be comprised within the scale of the ocular micrometer. For secondary reasons, however, the mirror micrometer was moved in several cases, but as the amount of shift is registered on the ocular micrometer, no essential discontinuity is introduced in this way.

In addition to the reading e, the very variable temperature of the laboratory was taken and the data are inserted in the lower curve, with the values increasing downward. With this arrangement the two curves show considerable general resemblance up to about February 10. One may suppose that

these are actual changes in the inclination of the pier as a result of non-uniform heating; if the viscous deformations of apparatus were accentuated with rise of temperature, the two curves should not be in opposition and there does not seem to be any means by which mere thermal expansion could produce this result.

After February 10 the effect of the temperature of the laboratory to be inferred in contrasting the two curves is no longer apparent. In its place, however, is an interesting indication of the effect of external meteorological conditions on the inclination of the pier. Thus, while the curve has in a general way been falling toward the time of the intense cold spell culmina-

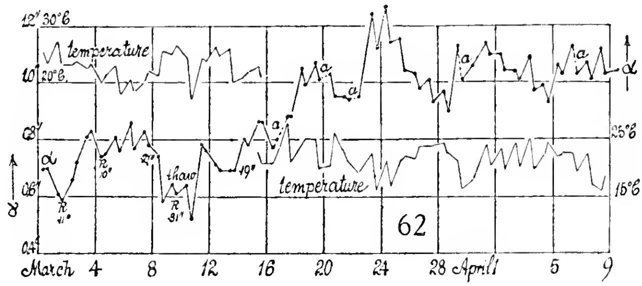


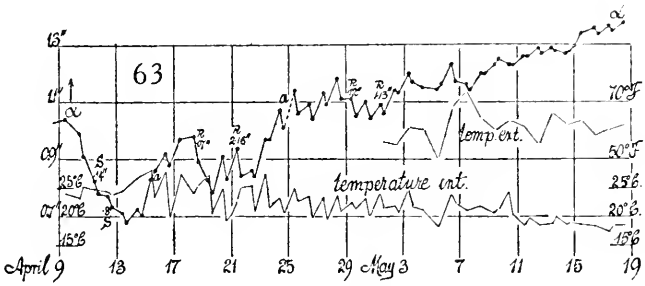
ting in February 5, the marked thaw following soon thereafter is accompanied by a rise of curve as far as February 18 and later. Then, with a second cold spell, the curve falls again to February 24 and in turn rises with the next thaw as far as February 27. It seems hardly probable that so marked a general behavior can be a mere coincidence, and I have regarded it probable that the hill on which the laboratory stands is undergoing similar inclinations.

In conclusion, therefore, one may regard the general trend of the curve upward as following from some yield within the parts of the apparatus. On this is superimposed the effect of the warping of the pier from internal causes, (chiefly change of temperature) and the change of inclination of the laboratory as a whole, due to external climatic conditions.

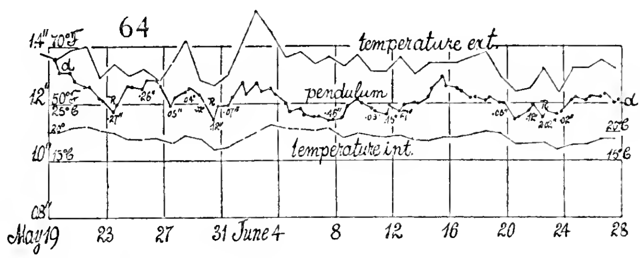
51. Observations continued.—The apparatus was now taken down for repairs and some modifications. A brace was added at t'' (fig. 57), supporting the end of the interferometer platform, an addition to which I was at first averse; but if thermal expansion is equal throughout the iron framework, it should not produce discrepancies. The new data (figs. 62 to 65) contain observations between February and August, a period of about 5 months. They are constructed in the same way as the preceding (fig. 61) and places where readjustment was needed are marked *a*. Moreover, the adjustment

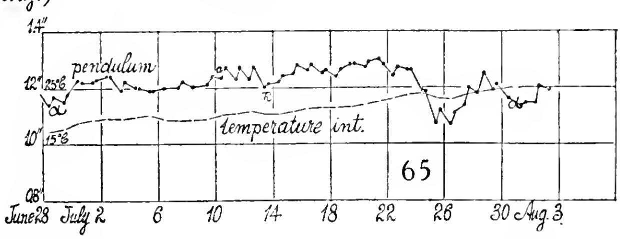
screws of the mirrors after March 16 were relieved from strain as far as possible, an operation which required several subsequent trials, after which the stability of adjustment (persistently coincident slit-images) was much improved. The (internal) temperature (degrees centigrade) of the laboratory





has also been added, here laid off positively upward. In May and June the external temperature, as reported by the Weather Bureau in degrees Fahrenheit, is inscribed, and the precipitation in inches (marked R) is often indicated.





If we examine the curves as a whole, the marked variability of the amount of inclination, α (seconds of arc), in March, April, and May, is in contrast with the relative quiescence in the latter part of May, in June and July.

This is at once an evidence of the discrepant effect, direct and indirect, of the heating of the laboratory. In a general way, moreover, rise of internal temperature is more apt to be associated with a fall of the α curve, though the similarity is far from consistent. In June and July the continuous general rise of temperature associates itself with a rise of the α curve. Thus the temperature relations, which undoubtedly exist, are very complicated, as is to be expected. During March and April the α values fluctuate between $\alpha = 0.6''$ and $\alpha = 1.3''$, but in May the curve rises and remains permanently above $\alpha = 1.0''$. In fact, this trend toward large α values may be said to begin in the middle of April. A comparison with the external temperatures (degrees Fahrenheit) leads to no consistent results. The effect of the thaw about March 9 is apparently well-marked, but the pocket of the α curve may here also be attributed to the ascent of the temperature curve. Subsequent thaws are not indicated. There are a number of rain-pockets in the α curve, but they do not bear a definite relation to the amount of precipitation. Rains in summer also usually involve changes of temperature. After the period of comparative quiescence of the α values in May, June, and July the fall recorded after July 24 is peculiar. Nothing local was detected to account for it and it is temporary.

So far as the observations were made to test the availability of the apparatus, they may be regarded as quite satisfactory, but it is obvious they can be used as evidence only during the summer months in the absence of internal heating and that marked temperature changes are menacing under all circumstances. I had hoped that this would not be the case. The conclusions already drawn above thus hold in the sequel. It is my purpose to install this sensitive apparatus under fit surroundings at some future opportunity. It should, then, contribute substantially to geophysical investigation.

CHAPTER V.

THE INTERFEROMETRY OF VIBRATING SYSTEMS.

52. Introductory.—The high luminosity of the achromatic interferences and the occurrence of but two sharp fringes make it possible to utilize them even in cases when the auxiliary mirrors vibrate. Experiments of a similar kind have been previously tried with telephones and the spectrum ellipses;[1] but these fringes do not easily admit of being drawn out into a ribbon and there is usually deficient light. The endeavor to distinguish the phases of vibrating telephonic systems was partially successful, but the marked importance of synchronism or resonance in these systems is the chief outcome of the research.

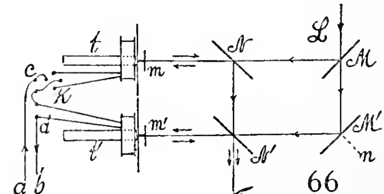
53. Telephonic apparatus.—I began the work with two similar telephones, as shown at t,t', in figure 66. Small mirrors were rigidly attached to the centers of the diaphragms and each of the telephones secured on a standard which admitted of adjustment around vertical and horizontal axes. The intermittent current was supplied at a,b, by a small induction coil with a rheostat in circuit. Four clamp-screws at c,d were available for putting the telephone bobbins in series or in parallel. One telephone could be reversed in action by the commutator K. White light L from a collimator was reflected or transmitted by the half-silvered mirrors M, M', N, N' of the interferometer and from m,m' on the telephones, as indicated by the arrows. M' was on a micrometer with the screw on the direction n. To facilitate the finding of the fringes one of the telephones, t', should also be on a micrometer with the screw normal to m'. The use of n requires special precautions stated below. The fringes when found are observed by the vibration telescope at T. It is sometimes difficult to catch the fringes even when using the spectro-telescope, owing to the accentuated quiver of the system, and the work is simplified by first supporting the diaphragm against vibration.

The vibration telescope is shown in vertical section in figure 67, with the ocular at E and the objective originally at e, the tube being supported on the standard d and clamp cc, admitting of raising and lowering and slight rotation around the horizontal axis b. The objective A has been removed and is now supported by a flat steel spring s,s, in front of its former position. Hence the ocular holder is a simple tube which can be thrust far inward and clamped in any position by three screws at a.

In order that the ocular may vibrate parallel to the fringes, and as these appear in all angles of altitude, the special vibratory system f,g, k,s has been devised. The rod reaching downward (about 10 to 15 cm. long) is attached

[1] Carnegie Inst. Wash. Pub. No. 149, Part III, 1914, pp. 208-213.

to a ring rr, capable of revolving with friction around the tube of the objective and of being fixed for any angle in altitude of the rod. The horizontal clamp g, adjustably attached to the rod f, supports the spring s. This passes through a vertical crevice in g and is fixed by the vertical set-screw p and the two oblique adjusting strips k on either side of g. These pieces, k, are clamped by the screws at h, and serve as holders of the two coaxial set-screws at i on either side of sk, so that the objective A may be centered relative to the tube Ee for any oblique position of f, which must be normal to the fringes. The vibration may frequently be considerably changed by sliding f and s in g and reclamping the system. If the objective is to be fixed, a screw at n may be depressed for the purpose. The vibration may be started and stimulated from time to time manually. It has not, thus far, been necessary to add an electromagnetic vibrator.



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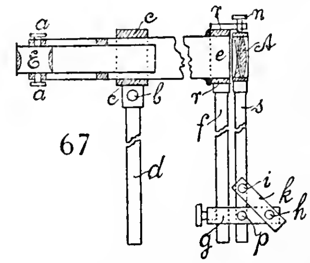


67

To find the fringes a spectro-telescope will usually first have to be used. This is then replaced by the apparatus in figure 67, for observation. If not too far displaced, the fringes of a vibrating system may thereafter be found by the vibration telescope even when they can not be seen with the fixed telescope, as they overlap during vibration. A fine slit, so long as it supplies sufficient light, gives the sharpest wave-curves. The angle of altitude of the fringes is of no consequence, since f is set in altitude normal to them; but it is of advantage to rotate the slit also, until its image in the telescope is nearly normal to the fringes.

Under all circumstances the two spots of light representing the slit, if caught objectively on a screen at T, must be nearly coincident, horizontally and vertically, when the rays at T are parallel. If these spots are too far apart the fringes will be very small or even absent. A search for them by any method is then useless. To meet this preliminary but essential condition, the spots are first made coincident by rotating N (horizontal and vertical axes) and thereafter rotating N' until the images coincide in the telescope at T. One or two adjustments of this kind usually suffice, since perfect coincidence is non-essential and in fact undesirable, because the fringes are then too large for convenience.

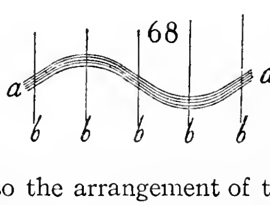
54. Observations.—The use of two telephones soon showed itself to be unsuitable for the present purposes; for the diaphragms oscillate not merely fore and aft, as is here desirable, but locally around horizontal and vertical axes as well, particularly when the vibration is relatively intense. Hence the coincident slit-images of the silent telephone periodically separate or pass

through each other when the diaphragms vibrate. This shows itself in a peculiar manner in the field of the vibration telescope, as indicated in figure 68. The whitish field carrying the fringe-waves *aa*, due to the fore-and-aft motion of the mirrors *m*, *m′* (fig. 66) on the diaphragms of the telephones, is intersected by nearly equidistant vivid lines *b*,*b*, normal to the waves. The latter are apt to be broken and appear only as traces between the vertical lines. The wave-length of *a*,*a* and the distance apart of *b*,*b* will depend on the maximum speed of the objective of the vibration telescope, but the distance *bb* and the form of *aa* usually have some simple relation to each other, so that frequently the wave form is lost entirely and merely oblique lines with the same inclination are seen between the lines *b*, *b*.

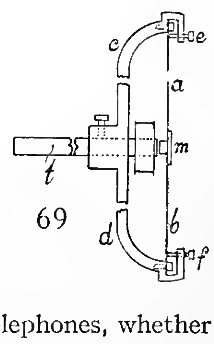
To account for these occurrences of the lines *b*, it is sufficient to recall that the originally coincident identical slit-images are in vibration through each other in some direction relative to the lengths of the slits, effectively therefore normal to this direction. The amount of this displacement is small, but it is greater than the breadth of the fine slit-images. Hence these images will be seen clearly only at the elongations when the slit-images are temporarily stationary and at a maximum distance apart. These apparitions are the lines *bb* and they belong to a system with higher frequency than the objective. Hence, also, the waves *a*, *a* (fig. 68) are absent at *b*,*b*; for here the slit-images are so far apart as to eliminate the interferences.

It made little difference how the two telephones were connected. In fact, one telephone may be thrown out of circuit. Vibration is thus communicated mechanically similarly to the case when the fingers drum lightly on the table. If a fine wire is drawn across the slit, the shadow remains straight unless the vibrations are very intense; then beating waves in trains run along the black line of shadow.

The telephone diaphragms were now removed and replaced by the two long strips of steel made from hack-saw blades 20 cm. long and about 1 cm. broad and 0.06 cm. thick, rigidly attached to the body of the telephone by the arch *cd* (fig. 69.) The mirror *m* was cemented to the middle of this strip, balanced by a piece of iron on the other side. To approach this as near the magnet as possible, forcing screws, *e* and *f*, were provided at a little distance from the end of the strip. In this case the vibration of the strip *ab* and the mirror *m* at its middle was fore and aft only, and as a consequence the lines *b*, *b′* in figure 69 vanished completely. Here also the arrangement of telephones, whether in series or parallel, made a decided difference in the amplitude of the waves *a*, which could be increased many times the breadth between successive fringes before the waves became turbulent and broke up.



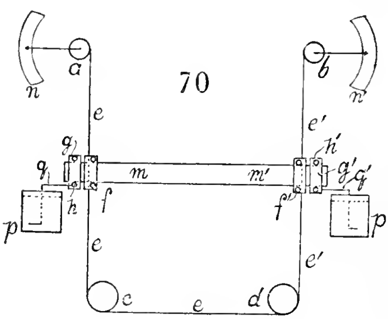
68

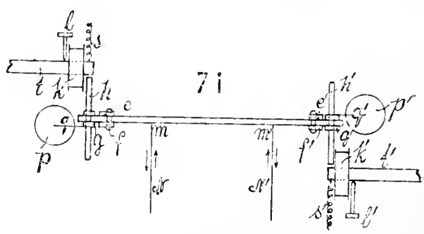


69

The telephonic system was now put in place of the galvanometer of a Wheatstone bridge, the small induction-coil being used as a source of current. It was then found that the adjusted resistances of the bridge could be changed by no more than a few tenths of 1 per cent before the even band of fringes changed appreciably to the wave-form *aa* (fig. 68). But as a dynamometer the instrument was still much inferior to the audible telephone. Currents of an order less than an average 10^{-4} ampere would be difficult to detect.

55. Bifilar systems.—To utilize such a system as figure 69 to full advantage it would be necessary to attune the springs *ab* of the two telephones to the same period, which should then be as nearly as possible identical with the period of the source of intermittent or alternating current. As an earth inductor or a small magneto inductor (single magnet rotating in a flat coil) would have to be used in the latter case, it seemed best to convert the apparatus into a bifilar vibrator as shown in figures 70 and 71 in elevation and plan. Here mm' is a strip of thin mirror plate-glass, about 32 cm. long, 1 cm. broad, and 2 mm. thick, horizontal and in a position to receive the rays NN' of the interferometer (compare fig. 66). Motion of mm, parallel to itself, fore and aft, will therefore produce no effect on the fringes; but any rotation around a vertical axis will be immediately apparent, as indicated in the above methods for small angles.





This strip of glass is supported by the bifilar system *ee*, $e'e'$, made of a single thin wire of brass, 0.2 mm. in diameter. The ends of ee' are wound around the horizontal screws a, b, which rotate with friction and are supplied with an index and scale n, n', so that any tension may be imparted to the wire. This passes below under the pulleys c, d, as nearly free from friction as possible, with the object of securing the same tension throughout ee'. Flat clamps ff', of fiber and screws, attach the strip *mm* to the wire at any height, but necessarily near the middle of the vertical threads, where it receives the rays NN'.

The telephonic system consists of the soft-iron horizontal screws hh', similarly attached to mm' by the flat fiber clamps g, g' and the telephones t, t' (omitted in fig. 70). These were made of large flat files, each provided near its end with an appropriate bobbin, kk', of fine telephone wire, the ends of which are attached to clamps, as already shown in figure 66, with one of the telephones provided with a commutator for reversing its current. The resistance of each bobbin was about 140 ohms. The screws ll' are used to ap-

proach the telephone magnets tt' as near the soft-iron armatures hh' as possible without overstepping the unstable position, in view of the tension of the tense wire e, e'. To prevent the sticking of h to t, which is very annoying, small rubber buffers may be placed between. It is not usually practicable t approach h to t by more than 1 or 2 mm. and keep h perfectly free. To give the vibrating system adequate damping, thin wires qq', less than a millimeter thick, bent, and dipping into lubricating oil in small vats p, p', suffice. To change the damping the latter may be lowered or even removed. The fibers e,e' were about 45 cm. long and their distance apart about 29 cm. Their period and that of the vibrating telescope were made about the same, on the average about 0.2 sec., and this was for convenience nearly the same as the period of the vibrating telescope and of the induced alternating current.

It is convenient to insert an extra telephone (resistance about 100 ohms) in circuit, in order to insure against breaks of contact or other discrepancy, when the perturbation of fringes ceases.

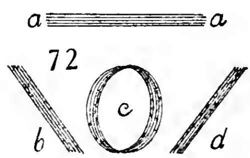
As a generator an earth inductor with a coil of wire 60 cm. in diameter was at first used. It was turned by a small motor, and by putting a sliding rheostat in circuit the period could be varied from about 0.19 to 0.26 second. To measure the average intensity of current a Siemens precision dynamometer was installed; but though indicating currents as low as even within one-tenth of an average milliampere, it was not influenced by the earth inductor, though at maximum speed. This was therefore replaced by a small magneto consisting of a bar-magnet about 6 inches long, rotating in an oblong coil. By aid of the rheostat and appropriate pulley-wheels large differences of speed could be obtained and maintained at any value, so that periods from about 0.1 to 0.3 sec. were available. With a period of 0.2 sec., moreover, it showed a deflection on the dynamometer and, being in general lighter and more easily controlled, was preferable to the earlier instruments.

The three vibrating systems (mirror, telescope, alternator) thus all admit of an adjustment of their periods, and these should be nearly the same if the elliptic system of Lissajous curves are to be obtained, which is the preferable case. A change of the tension of the wires ee' in figures 70 and 71, or any adjustments at the telephones, calls for a fresh search for fringes; but this is not difficult if the spectro-telescope is first used and the admonitions relative to the objective coincidence of pencils entering the telescope, as well as their parallelism, as above explained, are given consideration. These difficulties do not enter when the mirrors can be displaced normally to the incident rays.

In addition to the telephone tt' in figure 71, coils in great variety were used. The telephones were also placed within and without the rectangle of wires e,e' and in the same or on opposite (as in fig. 71) sides of mm'. But the phenomena in such cases were not dissimilar nor advantageous.

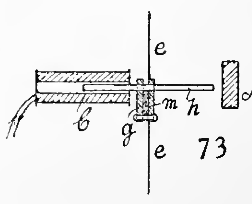
For general purposes the mass of the vibrating system mm' should be diminished, as it could easily be; but the straight blade of glass (in preference to two small mirrors) is a convenience in adjustment and here suffices.

56. Further observations.—Without synchronism in the two vibrating systems (current and telephone), the motion of fringes obtained is practically inappreciable when average currents within the order of milliamperes are treated. This is a curious and at first a disappointing result. As soon, however, as approximate synchronism is established, the sensitiveness of the apparatus increases enormously. It is best for this purpose to vary the period of the motor of the alternate-current generator by the slide rheostat. If the fringes are horizontal and the objective therefore vibrating horizontally across the vertical slit-image, the motion of the fringes is vertical. Hence the horizontal band *a* (fig. 72), in the absence of current, at once takes the form *b,c,d,c,b,* in succession, with the opposition of rotation in *c* quite visible. The continuous change of these may be most conveniently accelerated or retarded by controlling the motor of the alternator with the slide rheostat. Since the lines *b* and *d* are of different inclination, they will usually show a difference of breadth. Circles appear when the amplitude of the objective has sufficiently decreased, and it is advantageous, as a rule, to keep this amplitude small, for the phenomenon is then more luminous and brilliant. Very large fringes are not usually desirable, as they are too mobile and may pass out of the field. The smaller fringes are quite satisfactory and more easily obtained. When the tension of the fibers *e,e′* (fig. 70) is too small the fringes drift, showing that with the varying magnetization there is no persistent position of equilibrium. It is annoying if they leave the field while executing their gyrations, though they may always be restored by moving the micrometer. For mean tensions the higher Lissajous curves 2:3, 3:4 may be obtained, both for the alternator moving at smaller and at larger periods than the vibrating mirror. To obtain them the motor running at maximum speed is gradually slowed down by means of the rheostat, when the forms appear in succession, passing through the elliptic series at mean speeds. This in fact is the best tension for practical purposes. The size of the curves and the brilliancy of the whole display is increased by decreasing the damping or lowering the cups *p*, in figure 70.



72

A beautiful phenomenon is observed when the magnets *t,t′* hold the armatures *h,h′* to the intervening rubber cushions and the fringes are fairly large. The slightest vibration anywhere in the vicinity will then cause the even band *a* (fig. 72) to change to magnificent large roof-shaped or violin waves. This loose contact device could, no doubt, be made useful for observational purposes. Without the vibration telescope such fringes would not be visible, as they overlap during vibration.



73

The Lissajous curves continue to be very marked when additional resistances as high as 1,000 ohms are put into the circuit of the alternator. Indeed, they do not vanish appreciably, even for an additional 10,000 ohms, if well

produced. They do not, however, increase in size in proportion to the decrease of variable resistance, a result attributable to the amount of resistance and inductance necessarily in circuit, the effect of which is relatively large when the additional resistance is small.

To obtain some idea of the smallest average current appreciable, the Siemens dynamometer may be put in circuit, though unfortunately it has a resistance as high as about 1,000 ohms. With the magneto inductor running at a speed of $T=0.17$ sec., the Siemens showed a deflection of but 0.02 cm. on the given scale, owing to this resistance.

Estimating the average current i as $i=C\sqrt{\varphi}$, where C is the dynamometer constant and φ is the deflection in centimeters, the value of C as found by Clarke's cell and resistances was $C=5\times10^{-4}$, so that the average magneto current corresponding to $\varphi=0.02$ cm. is $i=5\times10^{-4}\times\sqrt{0.02}=7\times10^{-5}$ ampere. The resistance in circuit was here about 1,500 ohms. If an additional 10,000 ohms is inserted and the magneto reduced in speed to $T=0.25$ sec., the current is still appreciable at the interferometer and would be of the average value of $i=6\times10^{-6}$ ampere. Although no account has been taken of self-induction, it is improbable that the smallest average current here observable by the interferometer apparatus in the earlier form could have been much within a microampere. In this respect the device was somewhat disappointing.

In a later and more refined adjustment the ellipses obtained with an insertion 10,000 were found to fill (as to their vertical or current axes) fully one-quarter of the field of the telescope. As little as one-tenth to one-hundredth of this would be easily appreciable with certainty, so that the minimum average current capable of detection may be estimated as a few 10^{-7} amperes. It must be remembered, however, that the above mirrors (mm') and appurtenances are unnecessarily heavy, and the bifilar too robust. The improvement would therefore consist in constructing a very light needle and delicate bifilar.

If the dampers p,p' are removed, the ellipses, even at high resistance, are apt to pass out of the field of the telescope. Even if the principles of forced vibrations are applicable, the system in the absence of current is too mobile for convenience.

When the magneto was run at maximum speed (without pulleys) a deflection of about 0.12 cm. was obtained on the dynamometer, corresponding to an average current, therefore, of about $i=1.5\times10^{-4}$ ampere. In this case the even band a (fig. 72) takes the definite shape of a train of waves (fig. 68) of short wave-length, with amplitude of but 10 or 20 fringe-breadths. These also admit of the additional insertion of several thousand ohms before they are reduced to the linear band.

If one of the telephones is reversed, the fringes showing marked vibration (*bcd*, fig. 72) in the first position frequently cease to show any vibration (*a*, fig. 72) until the ellipses in the former case are very large (small resistance in circuit). The results in such a case are uniformly consistent. Thus, for

the commutator positions I and II and resistances R in circuit, the results were, for instance:

I	Circuit open.	II	R
Band	Band	Strong ellipses	100 ohms
Band	Band	Strong ellipses	500
Band	Band	Smaller ellipses	1,000
Band	Band	Ellipses just seen	2,000
Band	Band	Band	5,000

The reason of this is apparent, for when the telephones are joined in series the mirror mm' (fig. 70) is periodically rotated and released around a vertical axis and the displacement of fringes is proportional to the small angular amplitude of rotation. If, however, the telephones are connected differentially, the mirror mm', if properly adjusted, merely moves parallel to itself, fore and aft, and the fringes remain stationary. More usually, however, there is a difference in the size of ellipses (*cæt. par.*) in the two cases and at other times there is scarcely any difference at all appreciable. In such cases it seems probable that the periods of the two filaments ee and $e'e'$ on opposite sides of mm' are not the same, or differ in amplitude (attracting forces of different strengths), so that the fore-and-aft motion is accompanied by more or less rotation, residually. It is in fact difficult to make the two telephones, etc., quite identical in action.

It is for the investigation of this question that the adjustment pushing-screws l, l' and springs s, s' (pulling toward the rear), the telephones being on a vertical axes, were provided. If one of these, t for instance, is placed in a definite effective position, while t' is relatively far from its armature h', the screw l' may be gradually pushed forward, diminishing the distance to the minimum. The effect of this is further to rotate mm', and if the turning of l' is cautiously done the fringes may be passed from top to bottom of the telescopic field by l' and restored to position by the micrometer. After each step of the experiment the fringes are observed, when actuated by the alternator, in the two positions of the commutator of one telephone. In this way it is possible to find the adjustment in which for one position of the commutator there is excessive motion of fringes, whereas for the other there is practically no motion, as in the example above. If the distance (t', h') is markedly larger or smaller, the distinction of the two positions of the commutator is lessened and may even vanish.

There is usually some vibration figure (1/1, 3/4, etc.) best adapted for the given adjustment, even if the other figures appear. When this is chosen, the distance from magnet to armature (t' to h') makes little difference within reasonable distances. Such a result is not unexpected, for the whole phenomenon is relative, larger differences corresponding to larger total forces. The distance between magnet and armature (here on one side) does, however, affect the tension of the string, since the forces and the stretch are greater for smaller distances and the period therefore smaller. This is the simplest

method of obtaining unison on the two sides of the bifilar, and as the magnet t' is set by its adjustment screw l', the motion of the fringes while oscillating is seen in the field of the telescope and they are thus never lost. Regulation at a, b, figure 70, is more difficult.

Adjusting the bifilar as to tension in this way, there is one position or distance between h' and t' pretty sharply determinable for which the fringe bands change to stationary ellipses in the *absence* of all current. This peculiar result is at first puzzling, but since it is quite synchronous with the period of the telescope (stationary ellipse), it is obvious that the motion of the objective is the cause of the phenomenon and that the fibers are now in unison with its period. For distances h', t', greater or smaller, the ellipses soon return to bands. The effect of the alternating current on the stationary ellipse is very beautiful. It now oscillates very much like a smoke-ring for one commutator position, whereas it passes in an accentuated way through all phases for the other. Naturally, very complicated displays are also obtained in this double superposition; but practically the vibration of the telescope objective does not disturb the bifilar of the interferometer, unless under the exceptional condition of complete unison, even when both instruments are on the same (insulated) table, a convenience not at all necessary.

Utilizing the preceding adjustment giving ellipses and bands, respectively, in the two positions of the commutator, many experiments were made to detect a change of phase when a large inductance is placed on both sides of one of the telephones. But in none of the experiments thus far was any difference discernible to be attributed to the presence of the inductance. The endeavor to produce in part, by the mere insertion of inductance, an effect similar to commutation has not, therefore, been realized.

Among other promiscuous experiments I may refer to the use of a variety of telephones, single and bipolar; to changes in their position, sometimes with their poles (as at mm', fig. 70) between, and sometimes with the poles on the outside of the wire filaments ee, ee' (as drawn in the figure); to bifilars of thread instead of wire; to bifilars of watch-spring; to different sizes and kinds of armatures, etc. Coils, moreover, of different resistances and size of wire were tried, for instance, in figure 73 (which preserves the notation of fig. 70), where C is the coil, h the soft-iron (screw) armature, and M a strong inducing magnet. In the absence of M, no effect was obtained, even when C was provided with a core of soft iron reacting on h. In other words, a magnetizing system is inefficient. In the presence of the magnet M, however, the results were marked, but not better than the above, while the system itself is more complicated. The replacing of h by a hard-steel permanent magnet gave good results, but of inferior sensitiveness. To obviate the annoyances of contact between armature h and M, the latter might advantageously be replaced by a small magnetizing coil surrounding the free end of h and supplied separately with current. Endeavors were also made to utilize the repulsion between a permanent magnet at h and a similar pole at M. But M in such a case reversed the polarity of h.

As a tuned system responding to definite periods only, the vibration interferometer is quite sensitive, provided the average currents are of the order of several microamperes. Between the types of compound vibration-curves corresponding to frequency ratios of 4/3, 3/2, 1/1, 2/3, 3/4, there is usually an unbroken band of fringes. If the ratio of periods remains fixed, the vibration curve of course remains fixed, which is the usual sharp acoustic criterion. When the ellipses (out of tune) change continuously between lines of different inclination, the passage in one direction is often gradual, whereas in the reverse or return direction it is almost sudden. Linear forms flop into linear forms, as it were. No doubt this is related to the vibration of a bifilar system like the above, where the two ends are liable to vibrate alternately. When the average currents approach the order of 10^{-4} cm. the bands become sinuous for all periods not excessively high or low, as already stated.